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Determination of Areas and Basins of Attraction in Planar Dynamical Systems
using Meshless Collocation

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This work is focused on the approximation of sets of attractive solutions of planar dynamical systems. Existing work has shown that for many dynamical systems a Riemannian contraction metric can be used to determine sets of solutions with certain attraction properties. For autonomous dynamical systems in \mathbb{R}^2 it has been shown that the Riemannian contraction metric can be reduced to a scalar weight function W . In this work we show that a similar result holds true for finite-time dynamical systems with one spatial dimension. We show how meshless collocation can be used to construct an approximation of W . The approximated weight function can then be used to determine subsets of the area of exponential attraction. This is the first time a method has been introduced to approximate finite-time areas of exponential attraction. We also give a convergence proof for the method. For autonomous dynamical systems in \mathbb{R}^2 there already exists a method that uses W to determine a subset of the basin of attraction of an exponentially stable periodic orbit, Ω . However that method relies on properties of Ω being known. We show that the existing equation for W can be manipulated so that no knowledge of the periodic orbit is required to approximate W . We present a method that utilises meshless collocation to approximate W and show that the method is convergent. The approximant of W is then used to determine subsets of the basin of attraction of Ω .

Dedication

I would like to give my deepest thanks to my supervisor Peter, without his understanding, patience, kindness and support I would not have finished this work . I would also like to thank my partner Mel for her tireless support. Finally I'd like to thank my internal and external examiners for their feedback regarding the original submission.

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Chapter 1

Introduction

1.1 Overview

In this work we are interested in approximating sets of attractive solutions for planar dynamical systems. Attractive sets have long been a focus in dynamical systems. Lyapunov methods tend to be the dominant method for determining attractive sets in this setting. But over the past 10 years there has been a growing focus on using a local contraction metric to determine sets of attractive solutions. In this work we build upon these results. When restricted to planar systems the contraction property can be characterised by a scalar weight function W . We derive a partial differential equation with W as a solution. However these differential equations contain an unknown constant. By isolating the constant and taking the orbital derivative we obtain an equation where W is the only unknown. We present a methodology to numerically solve these equations via meshless collocation and show that the method is convergent. We then present numerical examples.

The first planar system we work with is on a finite-time interval with one spatial dimension. Finite-time dynamical systems are a recent yet rapidly growing area of study. In this work we use the local contraction property to derive a method of approximating subsets of the area of exponential attraction for nonautonomous finite-time differential equations. The area of exponential attraction is a relatively new concept. We show that for finite-time dynamical systems, a Riemannian contraction metric can be reduced to a scalar weight function. We derive a differential equation where the orbital derivative of W is the only unknown. We then present a method that numerically solves this equation using meshless

collocation and show that the method is convergent. This work adds to existing work of [Giesl, 2012] which uses Lyapunov functions to construct a subset of the domain of attraction of a solution. However that method is unable to determine whether a small neighbourhood of the solution is attractive. Our method does not have this limitation so can be used to fill in this missing information. It can also work on sets that either do not contain an equilibrium, or where it is not known if there is one. Areas of further study could extend this work to higher dimensions. The main contribution of this chapter is the introduction of a convergent method to approximate the area of exponential attraction.

The second planar system we work with is an autonomous differential equation in \mathbb{R}^2 . In this case we are interested in determining the basin of attraction of a periodic orbit, Ω . We build upon a method developed in [Giesl, 2007b] that used scalar weight functions to approximate the basin of attraction of Ω . However that method requires properties of Ω to be known a-priori. The method we present does not need any information about Ω to be implemented. We present a convergent method to approximate the weight function which can then be used to approximate the basin of attraction of a periodic orbit. The natural evolution for this method is to be extended into higher dimensions.

This work is structured as follows. In Section 1.2 we review the existing literature for finite-time dynamical systems and generalisations of Borg's criterion and the determination of the basin of attraction for periodic orbits.

Chapter 2 gives an introduction to meshless collocation, the method we use to numerically solve the partial differential equations. In Section 2.2 we show how it can be used to interpolate function values. This is extended in Sections 2.3 and 2.4 when we look at generalised and mixed interpolation respectively. In Section 2.5 we present the Wendland functions, a compactly supported radial basis function that we use in this work. Section 2.6 provides a brief overview of the basics of dynamical systems and then Section 2.7 gives an example of meshless collocation being used to solve a differential equation involving the orbital derivative. Then in Section 2.8 we give an overview of how the error estimates of the method are derived. Finally Section 2.9 reviews the existing literature that applies meshless collocation to dynamical systems.

In Chapter 3 we present our method for determining the area of exponential attraction for finite-time differential equations in one spatial dimension. In Section 3.2 we present some existing work on finite-time dynamics. We then go on to present a new result, in which we show how to derive a differential equation that can be used to approximate a scalar weight function, which can then be used to determine areas of exponential attraction. In Section 3.3 we show how the constant can be eliminated from the differential equation we derived in the previous section and prove that this new function can still be used to determine an area of exponential attraction. We then show how meshless collocation can be used to numerically solve the equation. In Section 3.3.2 we show that our method is convergent. Finally we close by demonstrating how the method works with two examples.

In Chapter 4 we present a method to determine the basin of attraction of a periodic orbit in \mathbb{R}^2 . Section 4.2 reviews existing results in the field. In Section 4.3 we derive a differential equation for the weight function that does not contain any extra unknown variables and show how this can be numerically solved by meshless collocation. In Section 4.4 we show that the method is convergent and in Section 4.5 we apply the method to two examples. We then end the work by drawing some concluding remarks in the discussion.

Throughout this work when the notation $\|\cdot\|$ is used it denotes the Euclidean norm.

1.2 State of the Art

We present an overview of existing research relating to the main areas this thesis covers: finite-time dynamical systems, Borg's criterion and the basin of attraction of periodic orbits. A literature review of meshless collocation in dynamical systems is given at the end of Chapter 2.

1.2.1 Finite-Time Dynamical Systems

Finite-time dynamical systems are a relatively new area of study. Despite many applications focusing on the behaviour of dynamical systems over a finite-time interval, there was not, until recently, an established theoretical framework for finite-time dynamics in the way there was for the classical asymptotic systems. In the early 2000s a mathematical theory of finite-time dynamics was introduced to help in the applications of fluid dynamics

and oceanography. The Lagrangian coherent structure (LCS) was first defined in [Haller, 2000]. An LCS is a manifold in a finite-time dynamical system. It describes a surface of trajectories that have a dominant influence on nearby trajectories and creates a coherent trajectory pattern. They can therefore be used to characterise attracting and repelling material surfaces. Along with the work on LCSs various diagnostic quantities were developed, such as finite-time Lyapunov exponents (FTLE), finite-time hyperbolicity and the finite-time spectrum. LCSs are a young but very active area of research and dominant in the field of finite-time dynamics. A brief review of LCSs can be found in [Peacock and Dabiri, 2010].

Since they were first defined, there has been active development of the theoretical concepts concerning LCSs, their detection, and the determination and classification of the type of trajectories they contain, e.g. hyperbolic, elliptic. The original work was built upon in [Haller, 2001] where existence criteria for LCSs in three dimensions were given. The n -dimensional case was then discussed in [Lekien and et al., 2007].

The concept of FTLEs is similar to that of the rate of exponential attraction, that is defined and studied in Chapter 3, except rather than being restricted to the behaviour of trajectories at the end time they vary throughout the time domain. A lot of the early work by Haller introducing LCSs also developed the theory of FTLEs. Further developments include [Shadden et al., 2005] which developed stronger links between the concepts of the LCSs and FTLEs and showed that through this link LCSs can be thought of as an invariant manifold and serve as boundaries of attractive areas. The link between certain FTLEs and hyperbolic LCSs is given in a rigorous mathematical framework in [Karrasch and Haller, 2013].

Another important tool in the field of finite-time dynamics and its application to fluid dynamics and oceanography is that of finite-time hyperbolicity, which originates from [Haller, 2000]. The classical definition is that a solution is hyperbolic if its linearisation has an exponential dichotomy, see [Katok and Hasselblatt, 1997]. The finite-time version of hyperbolicity is very similar and is defined in [Berger, 2011]. That paper goes on to provide a condition for finite-time hyperbolicity which can be used to show the existence of invariant manifolds. This is extended in [Berger, 2010], which shows that finite-time

hyperbolicity persists under small continuous perturbations. In [Duc and Siegmund, 2008] hyperbolicity and invariant manifolds are extended to 2-dimensional finite-time dynamical systems. Furthermore it gives definitions of finite-time stable and unstable manifolds. [Duc and Siegmund, 2011] builds upon the existing theories of hyperbolicity and applies them to a planar Hamiltonian flow. In [Berger et al., 2008] the authors partition the phase space into attracting, repelling, elliptic and hyperbolic regions and show that in the case of a linear system with constant coefficients this definition reduces to the traditional asymptotic definition. A unified theory of finite-time hyperbolicity is presented in [Doan et al., 2012]. This helps to unify existing theories of FTLEs and different theories of finite-time hyperbolicity that were introduced in [Rasmussen, 2010] and [Berger et al., 2009].

In [Rasmussen, 2007] weaker concepts of attraction in finite-time are introduced. The author then further develops these in [Rasmussen, 2010]. In that work a definition of attraction is given that allows for trajectories to not be attractive at every point, this differs to the concept of attractivity that comes out of the LCS school. It also introduces a finite-time version of the exponential dichotomy that leads to a finite-time spectrum, this differs from the finite-time spectrum given in [Berger et al., 2009]. Finally it also introduces a theory of finite-time bifurcation. The weaker concepts of finite-time attractivity are further extended in [Giesl and Rasmussen, 2012] where they are used to characterise finite-time areas of (exponential) attraction and domains of attraction of a solution. The work in Chapter 3 presents a method to approximate areas of exponential attraction via meshless collocation. Furthermore, [Giesl and Rasmussen, 2012] also introduces a finite-time Borg criterion and finite-time Lyapunov functions. Domains of attraction of a solution are approximated via finite-time Lyapunov functions and meshless collocation in [Giesl, 2012]. Finite-time dynamics are starting to move into a varied number of systems. In [Nersesov and Haddad, 2008] concepts of finite-time stability are extended to impulsive systems and [Kanno and Uchida, 2014] extends the FTLE theory into time-delayed systems.

1.2.2 Borg's Criterion and Approximation of the Basin of Attraction of Periodic Orbits

We consider the autonomous ordinary differential equation

$$\dot{x} = f(x), \tag{1.1}$$

with $x \in \mathbb{R}^n$ and $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ with $\sigma \geq 1$. Then Borg's criterion (Theorem 1.2.1) is a local contraction property first shown in [Borg, 1960].

Theorem 1.2.1. *Let $\emptyset \neq K \subset \mathbb{R}^n$ be a compact, connected and positively invariant set which contains no equilibrium. Moreover assume $L(p) < 0$ for all $p \in K$ where*

$$L(p) := \max_{\|v\|=1, v \perp f(p)} L(p, v),$$

$$L(p, v) := \langle Df(p)v, v \rangle.$$

Then there exists one and only one periodic orbit $\Omega \subset K$, which is exponentially stable. Moreover its basin of attraction $A(\Omega)$ contains K .

The theorem shows that if the distance between trajectories is decreasing at all points of their orbits then it is a sufficient condition for a periodic orbit to exist. However if trajectories increase their distance between each other for part of their orbit but then reduce it by an even greater amount on other parts there will still be a periodic orbit in K , these types of trajectories will be missed by the original Borg's criterion. Borg's criterion was generalised so that it incorporates a Riemannian contraction metric. This allows for local trajectories to increase in distance for part of their orbits as long as that increase is made up for by greater contraction at other parts of their orbits. This was first shown in [Stenström, 1962], where Borg's criterion was extended to include Riemannian metrics on Riemannian manifolds, for both equilibrium points and periodic orbits. However \mathbb{R}^n was treated as a special case in which the Riemannian metric becomes constant. In [Hartman and Olech, 1962] it is generalised by an integral function along the orbit, however this method was not computable. A similar method was used in [Leonov et al., 1995] to study the stability and instability of solutions, that are not necessarily periodic, by means of a linearisation that measures the distance between neighbouring trajectories on a linear area element transversal to one of the trajectories. In [Leonov et al., 2001] sufficient

conditions for the stability of differential equations on a Riemannian manifold are given by using a Lyapunov-type function and singular values. It is shown in [Giesl, 2004a] that a Borg function which is generalised with a Riemannian contraction metric is both a sufficient and necessary condition for the existence of a unique exponentially stable periodic orbit in a compact set. This result is then extended to determine the limit cycles for periodic differential equations in [Giesl, 2004b], non smooth systems in [Giesl, 2005] and almost periodic differential equations in [Giesl and Rasmussen, 2008]. [Giesl, 2015] extends the Riemannian contraction metric to work with an equilibrium point and shows it is a necessary and sufficient condition for a point to be exponentially stable and gives converse theorems that can aid in the construction of the metric.

In [Giesl, 2007b] a method to approximate the basin of attraction via meshless collocation is given, we extend this work in Chapter 4, where a deeper analysis is presented. This method was then extended to higher dimensions [Giesl, 2009] where the local contraction metric method is used in the neighbourhood of the periodic orbit and a Lyapunov function is used to determine attractivity outside that neighbourhood. In [Giesl and Hafstein, 2013] a method for approximating the basin of attraction of a periodic orbit for periodic differential equations in \mathbb{R}^n is given. They show that a Riemannian metric can be used to construct a symmetric matrix. That matrix being negative definite is a sufficient and necessary condition to characterise the basin of attraction. They use continuous piecewise affine functions to construct a Riemannian contraction metric and hence approximate the basin of attraction.

Chapter 2

Meshfree Collocation Applied to Dynamical Systems, using Radial Basis Functions (RBF)

2.1 Introduction

In both Chapters 3 and 4 of this work we use a specialised Borg criterion to approximate areas and basins of attraction. In both cases we derive a linear partial differential equation that cannot usually be solved analytically. The method we present to solve these equations numerically is meshless collocation. In particular we use radial basis functions. They have two main advantages. Firstly they are a meshfree method, this means there are no requirements on mesh triangulation. This allows grid points to be added easily wherever a finer granularity of data points is required. Secondly their error estimates are given in terms of the differential operator which aid the construction of error bounds.

In this chapter we give an overview of the methodology. Section 2.2 starts by describing how the technique is used to interpolate functions. This was how the methodology was originally employed. In Section 2.3 we describe its use to solve linear differential equations, this is the method that is most utilised in this work. In Section 2.4 we show how meshless collocation is used to solve a boundary value problem. We introduce the Wendland functions in Section 2.5, they are a compactly supported radial basis functions that we exclusively use in our numerical examples. Some background information about

dynamical systems is given in Section 2.6. An example of generalised interpolation applied to the orbital derivative is given in Section 2.7. In Section 2.8 we introduce the native space and describe how the error bounds are obtained. Finally in Section 2.9 we look at how meshless collocation has been used in the literature to approximate properties of dynamical systems. The work in this chapter is a brief introduction to the subject. For a more in depth analysis of radial basis functions we point the reader towards [Wendland, 2005] and [Buhmann, 2003]. For a deeper look at the use of radial basis functions in the construction of Lyapunov functions we suggest the book [Giesl, 2007a]. For a recent review of the computational approximation of Lyapunov functions see [Giesl and Hafstein, 2015].

2.2 Interpolation

Radial basis functions were traditionally used for scattered data approximation via function interpolation, examples of their use include surface reconstruction [Carr et al., 2001] and image compression [Boopathi and Arockiasamy, 2012]. In our study of radial basis functions we limit ourselves to functions that are positive definite, there are others such as conditionally positive definite functions. For more details on this please see [Wendland, 2005]. A function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is called radial if a function $\psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ exists such that $\Psi(x) = \psi(\|x\|)$. We now give a definition of a positive definite function, this is an important property of radial basis functions as it shows that matrices formed from them are positive definite and therefore invertible.

Definition 2.2.1. *A continuous function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is called positive-definite if for all $N \in \mathbb{N}$, all sets of pairwise distinct centres $X = \{x_1, \dots, x_N\}$ with each $x_i \in \mathbb{R}^n$, and all $\alpha \in \mathbb{R}^N \setminus \{0\}$*

$$\sum_{j=1}^N \sum_{k=1}^N \alpha_j \alpha_k \Psi(x_j - x_k) > 0,$$

i.e. the quadratic form is positive.

There are many examples of radial basis functions such as the Gaussians, multi-quadric and thin-plate splines. However we limit our focus to the compactly supported Wendland functions that were introduced in [Wendland, 1998]. These functions have compact support which means their interpolation matrices are sparse and therefore more numerically

stable and cheaper to compute. We introduce these in more detail in Section 2.5. Given a set of pairwise distinct points $X_N = \{x_1, \dots, x_N\}$, with each $x_k \in \mathbb{R}^n$, $k = 1 \dots N$ and a function $W : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $W(x_k)$ is known for each $k = 1, \dots, N$, a positive definite radial basis function can be used to construct an approximant $w : \mathbb{R}^n \rightarrow \mathbb{R}$, to W . We define the ansatz with respect to the grid X_N as

$$w(x) := \sum_{k=1}^N \alpha_k \Psi(x - x_k). \quad (2.1)$$

The coefficients for the ansatz are determined by solving a linear system where the true function W is equated to the ansatz w at each of the grid points. For each grid point $x_i \in X_N$ we have the following

$$w(x_i) = \sum_{k=1}^N \alpha_k \Psi(x_i - x_k) = W(x_i).$$

It is then possible to construct a linear system of the following form

$$A\alpha = W_{X_N}.$$

Where $A \in \mathbb{R}^{N \times N}$ is our interpolation matrix and has i^{th} , k^{th} entry $A_{ik} = \Psi(x_i - x_k) = \psi(\|x_i - x_k\|)$. The vector $W_{X_N} \in \mathbb{R}^{N \times 1}$ has i^{th} entry equal to the function W evaluated at x_i i.e. $W_{X_N i} = W(x_i)$. Finally $\alpha \in \mathbb{R}^{N \times 1}$ is an unknown coefficient vector. We determine α by solving the linear system. It is solvable due to the interpolation matrix being positive definite. This result is given in Lemma 3.3 from [Giesl, 2007a] which is shown below.

Lemma 2.2.1. *Let $\Psi(x)$ be a radial and positive definite function. Then for all grids X_N with pairwise distinct points and all functions W the corresponding interpolation problem has a unique solution. The interpolation matrix A is symmetric and positive definite.*

With the coefficients of w determined it can thus be used to approximate $W(x)$. The convergence result for function interpolation can be found in [Wendland, 2005].

2.3 Generalised Interpolation

In the 1990s radial basis functions emerged as a method for solving partial differential equations. The meshfree property of radial basis functions was particularly attractive

in this field where the incumbent methods such as finite differences and finite elements required the building of grids that complied to strict rules that made them more difficult to apply to unusual domains or geometries and expensive to refine. The initial advancement in the field was made by Kansa in [Kansa, 1990a] and developed in [Kansa, 1990b] where he applied the method to an elliptic, parabolic and hyperbolic Poisson problem. The method Kansa introduced has become known as unsymmetric collocation, this is because when the differential operator is applied to the ansatz the interpolation matrix is usually not symmetric and not guaranteed to be invertible. This method has been widely used in applications such as solving the shallow water equations on the sphere [Flyer and Wright, 2009]. Later in the 1990s a number of papers examined Hermite-Birkhoff interpolation using radial basis functions [Narcowich and Ward, 1994], [Iske, 1995], [Sun, 1994] and [Wu, 1992], this was the introduction of what is now known as symmetric collocation. In symmetric collocation the differential operator is applied in the construction of the ansatz before it is then again applied to the ansatz. This leads to a doubling of the order of the number of derivatives that need to be calculated in the collocation matrix but comes with the advantage that the matrix is symmetric. Symmetric collocation has been applied to a wide variety of problems such as Darcy's problem [Schröder and Wendland, 2011] and solving reaction diffusion problems on the surface of the sphere [Wendland, 2013]. A comparison of the two methods can be found in [Power and Barraco, 2002].

In this section we are interested in showing how radial basis functions can be used in symmetric collocation to give a solution to a linear partial differential equation of the form

$$\mathcal{D}W = f \text{ on } \Omega, \quad (2.2)$$

where Ω is a domain in \mathbb{R}^n and \mathcal{D} is a general linear operator of the form

$$\mathcal{D}W(x) = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha W(x), \quad (2.3)$$

where the coefficients have a certain smoothness, $c_\alpha \in C^\sigma(\overline{\Omega}, \mathbb{R})$ and $m, \sigma \in \mathbb{N}$. Let X_N be a set of pairwise distinct points $X_N = \{x_1, \dots, x_N\} \subset \mathbb{R}^n$ where the value $\mathcal{D}W(x_k)$ is known for all $x_k \in X_N$. We introduce Dirac's δ -operator, which is a pointwise operator, i.e.

$\delta_{x_k} W(x) = W(x_k)$. The following notation is used throughout this work $(\delta_{x_k} \circ \mathcal{D})^x W(x) = \mathcal{D}W(x_k)$, where the superscript x shows that the operator is to be applied with respect to the variable x . The reconstruction w of W with respect to the grid X_N and the operator \mathcal{D} uses the following ansatz

$$w(x) = \sum_{k=1}^N \alpha_k (\delta_{x_k} \circ \mathcal{D})^y \Psi(x - y).$$

We determine the coefficient vector by solving a linear system where we equate $\mathcal{D}w(x_i) = \mathcal{D}W(x_i)$ for each $x_i \in X_N$. This gives the following system

$$A\alpha = W_{\mathcal{D}, X_N}, \quad (2.4)$$

where $\alpha \in \mathbb{R}^{N \times 1}$ is the coefficient vector which is to be determined by solving the linear system and $W_{\mathcal{D}, X_N} \in \mathbb{R}^{N \times 1}$ is vector where the i^{th} entry is given by $W_{\mathcal{D}, X_N, i} = \mathcal{D}W(x_i) = f(x_i)$. Here $A \in \mathbb{R}^{N \times N}$ is our collocation matrix, it is positive definite under mild conditions on the points X_N and Ψ . The matrix has the following form

$$A = \begin{pmatrix} (\delta_{x_1} \circ \mathcal{D})^x (\delta_{x_1} \circ \mathcal{D})^y & \cdots & (\delta_{x_1} \circ \mathcal{D})^x (\delta_{x_N} \circ \mathcal{D})^y \\ \vdots & \ddots & \vdots \\ (\delta_{x_N} \circ \mathcal{D})^x (\delta_{x_1} \circ \mathcal{D})^y & \cdots & (\delta_{x_N} \circ \mathcal{D})^x (\delta_{x_N} \circ \mathcal{D})^y \end{pmatrix} \Psi(x - y).$$

Once α has been determined by solving the system (2.4) one can approximate $W(x)$ with $w(x)$ and $\mathcal{D}W(x)$ with $\mathcal{D}w(x)$. We discuss the convergence of this method in Section 2.8.

2.4 Mixed Interpolation

In Chapter 3 the partial differential equation we need to solve is a Dirichlet boundary problem. This requires both generalised interpolation for the interior points and interpolation for the boundary points. We now describe this method. Consider a boundary value problem

$$\begin{aligned} \mathcal{D}W &= f \text{ on } \Omega, \\ W &= g \text{ on } \Gamma. \end{aligned} \quad (2.5)$$

With Ω a domain in \mathbb{R}^n and $\Gamma \subset \partial\Omega$ i.e. Γ is a part of the boundary of Ω . \mathcal{D} is of the same form as (2.3). Consider a set of pairwise distinct points $X_N = \{x_1, \dots, x_N\}$ with $X_N \subset \Omega$ and a set of pairwise distinct points on the boundary $X_\Gamma = \{\eta_1, \dots, \eta_M\}$ with $X_\Gamma \subset \Gamma$. Then the ansatz to approximate W of (2.5) with respect to the grid $X_N \cup X_\Gamma$ is given by

$$w(x) = \sum_{k=1}^N \alpha_k (\delta_{x_k} \circ \mathcal{D})^y \Psi(x - y) + \sum_{k=1}^M \beta_k (\delta_{\eta_k})^y \Psi(x - y).$$

Then by equating $\mathcal{D}w(x_i) = f(x_i)$ for each $x_i \in X_N$ and $w(\eta_i) = g(\eta_i)$ for each $\eta_i \in X_\Gamma$ we construct the following collocation system which can be solved to determine the coefficient vectors α_i and β_i

$$\begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \quad (2.6)$$

With the following entries

$A \in \mathbb{R}^{N \times N}$ with the i^{th} , k^{th} entry $a_{ik} = (\delta_{x_i} \circ \mathcal{D})^x (\delta_{x_k} \circ \mathcal{D})^y \Psi(x - y)$

$C \in \mathbb{R}^{N \times M}$ with the i^{th} , k^{th} entry $c_{ik} = (\delta_{x_i} \circ \mathcal{D})^x (\delta_{\eta_k})^y \Psi(x - y)$,

$B \in \mathbb{R}^{M \times M}$ with the i^{th} , k^{th} entry $b_{ik} = (\delta_{\eta_i})^x (\delta_{\eta_k})^y \Psi(x - y) = \Psi(\eta_i - \eta_k)$,

$\alpha \in \mathbb{R}^{N \times 1}$ with i^{th} entry α_i ,

$\beta \in \mathbb{R}^{M \times 1}$ with i^{th} entry β_i ,

$F \in \mathbb{R}^{N \times 1}$ with i^{th} entry $F_i = f(x_i)$,

$G \in \mathbb{R}^{M \times 1}$ with i^{th} entry $G_i = g(x_i)$.

It is shown in Proposition 3.29 of [Giesl, 2007a] that a special case of the matrix $\begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$ is positive definite, hence the system is solvable. Further, as long as a sufficiently small grid is used w will be a good approximation of W .

2.5 Wendland Functions

In this work we use Wendland functions as our radial basis function. Wendland functions are piece-wise polynomial compactly supported functions. To see their derivation we point the reader to [Wendland, 1998]. The Wendland functions are supported on the range $[0, 1]$, but it is possible to scale the support radius so that it is either larger or smaller. A

smaller support radius leads to the interpolation matrix being sparser and results in better computational efficiency and numerical stability. However if the support is too small, relative to the fill distance and the problem being interpolated, then the approximation will not be good. Hence there is a trade off between computational complexity and accuracy that needs to be balanced through the choice of the support radius and the grid used for the approximation. Before defining the Wendland function we introduce a few points of notation. Let x be a real number. Then the truncated function $(x)_+$ is defined to be zero, if $x < 0$ and x otherwise. The floor function $\lfloor x \rfloor$ gives the largest integer i with $i \leq x$. We define the Wendland functions by

Definition 2.5.1. (*Wendland functions*). Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. We define the Wendland functions by

$$\psi_{d,k} := \mathcal{I}^k \psi_{\lfloor \frac{n}{2} \rfloor + k + 1}, \quad (2.7)$$

where $\psi_l := (1 - r)_+^l$ and $(\mathcal{I}\psi)(r) = \int_r^\infty t\psi(t)dt$ for all $r \in \mathbb{R}_0^+$.

Table 2.1 shows some examples of Wendland functions for different spatial dimensions. The table comes from [Wendland, 2005]. The notation \doteq denotes equality up to a positive constant factor.

Table 2.1: The Wendland functions.

Space Dimension	Function	Smoothness
d=1	$\psi_{1,0}(r) = (1 - r)_+$	C^0
	$\psi_{1,1}(r) \doteq (1 - r)_+^3 (3r + 1)$	C^2
	$\psi_{1,2}(r) \doteq (1 - r)_+^5 (8r^2 + 5r + 1)$	C^4
$d \leq 3$	$\psi_{3,0}(r) = (1 - r)_+^2$	C^0
	$\psi_{3,1}(r) \doteq (1 - r)_+^4 (4r + 1)$	C^2
	$\psi_{3,2}(r) \doteq (1 - r)_+^6 (35r^2 + 18r + 3)$	C^4
	$\psi_{3,3}(r) \doteq (1 - r)_+^8 (32r^3 + 25r^2 + 8r + 1)$	C^6
$d \leq 6$	$\psi_{6,4}(r) \doteq (1 - r)_+^{10} (2145r^4 + 2250r^3 + 1050r^2 + 250r + 25)$	C^8

In Table 2.2 we give the Wendland function $\psi_{6,4}$ which is the Wendland function that we predominately use in Chapters 3 and 4. We show a plot of $\psi_{6,4}$ in Figure 2.1.

Table 2.2: The table shows the Wendland function $\psi_{6,4} =: \psi_0$ and ψ_1 to ψ_4 , defined recursively by $\psi_{k+1}(r) := \frac{\partial_r \psi_k(r)}{r}$ for $k = 0, 1, 2, 3$. The support radius is taken to be 1.

	$\psi_{6,4}(r)$
$\psi_0(r)$	$(1-r)_+^{10}(25 + 250r + 1050r^2 + 2250r^3 + 2145r^4)$
$\psi_1(r)$	$-130(1-r)_+^9(5 + 45r + 159r^2 + 231r^3)$
$\psi_2(r)$	$17,160(1-r)_+^8(1 + 8r + 21r^2)$
$\psi_3(r)$	$-514,800(1-r)_+^7(1 + 7r)$
$\psi_4(r)$	$28,828,800(1-r)_+^6$

2.6 Introduction to Dynamical Systems

In this section we introduce some background theory for dynamical systems. For a more in depth look at the field see [Hartman, 2002]. Consider the the autonomous ordinary differential equation

$$\dot{x} = f(x), \quad (2.8)$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, i.e. $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable. The initial value problem of (2.8), $x(0) = \xi$ has a unique solution denoted by $x(t, \xi)$. We assume that solutions exist for all $t \geq 0$. We define a flow operator below,

Definition 2.6.1. (*Flow operator*) Define the flow operator S_t by $S_t \xi := x(t, \xi)$, where $x(t, \xi)$ is the solution of the system (2.8) with initial value $x(0) = \xi$ for all $t \geq 0$.

In this work we are interested in determining sets of solutions that are either attracted to each other (Chapter 3) or attracted to an invariant set (Chapter 4). Assume that (2.8) has an equilibrium at the point $x_0 \in \mathbb{R}^n$ such that $f(x_0) = 0$. Below we give the classical definitions of stability and attractivity of an equilibrium. Stability can be intuitively thought of as a constraint that once a trajectory is close to a point it does not then go far away from it. A solution is attractive to a point or set if it gets arbitrarily close to the point / set as time goes to infinity.

Definition 2.6.2. (*Stability and Attractivity*) Let x_0 be an equilibrium.

- x_0 is called stable, if for all $\epsilon > 0$ there is a $\delta > 0$ such that $S_t x \in B_\epsilon(x_0) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < \epsilon\}$ holds for all $x \in B_\delta(x_0)$ and all $t \geq 0$.

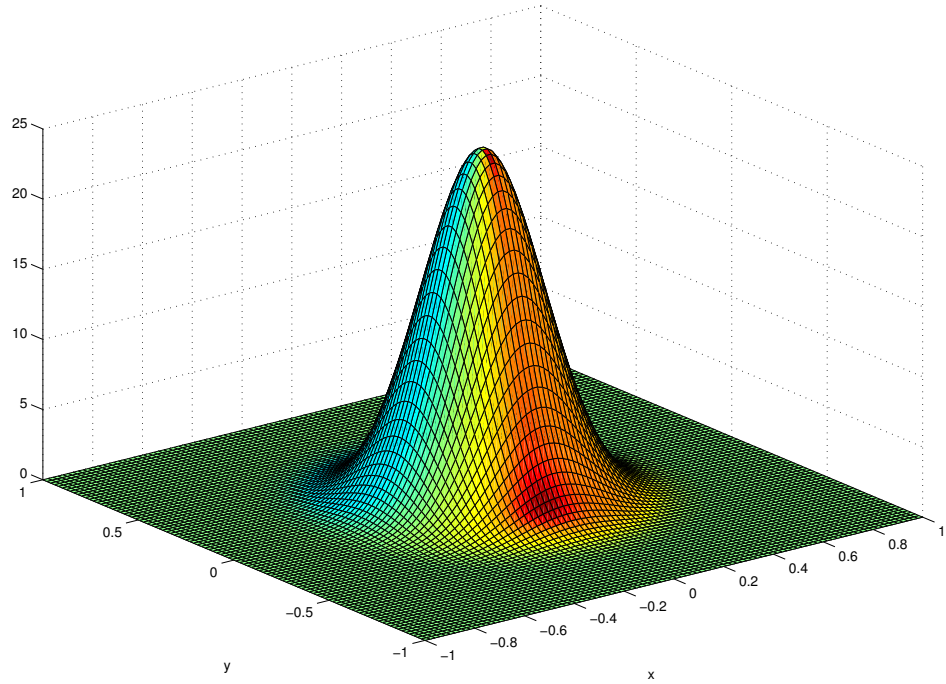


Figure 2.1: The Wendland function $\psi_{6,4}(r)$ with a support radius of one.

- x_0 is called *attractive*, if there is a $\delta' > 0$ such that $\|S_t x - x_0\| \xrightarrow{t \rightarrow \infty} 0$ holds for all $x \in B_{\delta'}(x_0)$.
- x_0 is called *asymptotically stable*, if x_0 is stable and attractive.
- x_0 is called *exponentially stable* (with exponent $-\nu < 0$), if x_0 is stable and there is a $\delta' > 0$ and $C \geq 1$ such that $\|S_t x - x_0\| \leq C e^{-\nu t} \|x - x_0\|$ holds for all $t \geq 0$ and $x \in B_{\delta'}(x_0)$.

For an asymptotically stable equilibrium x_0 there is a neighbourhood $B_{\delta'}(x_0)$ which is attracted by x_0 for $t \rightarrow \infty$. The set of all points that are eventually attracted to x_0 is called the basin of attraction of x_0 .

Definition 2.6.3. (*Basin of attraction*). The basin of attraction of an asymptotically stable equilibrium x_0 is defined as

$$A(x_0) := \{x \in \mathbb{R}^n \mid S_t x \xrightarrow{t \rightarrow \infty} x_0\}.$$

The long term evolution of the solution is determined by the ω -limit set.

Definition 2.6.4. (*ω -limit set*). We define the ω -limit set for a point $x \in \mathbb{R}^n$ with respect to (2.8) by

$$\omega(x) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S_t x}.$$

We have the following characterization of limit sets: $w \in \omega(x)$ if and only if there is a sequence $t_k \xrightarrow{t \rightarrow \infty} \infty$ such that $S_{t_k} x \xrightarrow{t \rightarrow \infty} w$.

Definition 2.6.5. (*Positively invariant*). A set $K \subset \mathbb{R}^n$ is called positively invariant if $S_t K \subset K$ holds for all $t \geq 0$.

In dynamical systems it is often necessary to understand how functions evolve along solutions, to describe this behaviour we use the orbital derivative. It is the gradient of the function projected onto the velocity at the point.

Definition 2.6.6. (*Orbital derivative*). The orbital derivative of a function $Q \in C^1(\mathbb{R}^n, \mathbb{R})$ with respect to (2.8) at a point $x \in \mathbb{R}^n$ is denoted by $Q'(x)$. It is defined by

$$Q'(x) = \langle \nabla Q(x), f(x) \rangle.$$

If the system is nonautonomous then the orbital derivative becomes,

$$Q'(x) = Q_t(x) + \langle \nabla Q(x), f(x) \rangle.$$

Lyapunov functions have a local minimum at an asymptotically stable equilibrium point and will decrease along solutions. The set on which the Lyapunov function is defined will be subset of the basin of attraction. The Lyapunov function has to decrease along solutions to ensure that its energy dissipates as it tends towards the solution. We define a Lyapunov function below taking the version given in [Giesl, 2007a]. It varies from the more common definition of Lyapunov functions in that it does not explicitly require x_0 to be the minimum of Q on K , as if K is a sublevel set of Q i.e. $K = \{x \in B \mid Q(x) \leq R^2\}$ the condition holds cf. Theorem 2.24 [Giesl, 2007a].

Definition 2.6.7. (*Lyapunov functions*). Let x_0 be an equilibrium of (2.8). Let $B \ni x_0$ be an open set. A function $Q \in C^1(B, \mathbb{R})$ is called a Lyapunov function if there is a set

$K \subset B$ with $x_0 \in \overset{\circ}{K}$, such that

$$Q'(x) < 0 \text{ for all } x \in K \setminus \{x_0\}.$$

The existence of a Lyapunov function implies the stability of the equilibrium. Furthermore, compact sublevels of the Lyapunov function which are completely contained in B are subsets of the basin of attraction of x_0 .

2.7 Example

In this example we give a little more detail of using radial basis functions in a generalised interpolation problem. Consider the system given in (2.8) and let x_0 be an equilibrium point of the system. We use the orbital derivative as our differential operator, \mathcal{L} i.e. $\mathcal{L}V(x) := V'(x)$. Let the partial differential equation we wish to numerically solve be

$$V'(x) = g(x).$$

Let $X_N = \{x_1, \dots, x_N\} \subset \mathbb{R}^n$ be a set of pairwise distinct points with $x_0 \notin X_N$. Let $\Psi(x) := \psi(\|x\|)$ be a Wendland function such that $\Psi \in C^2(\mathbb{R}^n, \mathbb{R})$. The ansatz for the approximation will be

$$\begin{aligned} v(x) &= \sum_{k=1}^N \alpha_k (\delta_{x_k} \circ \mathcal{L})^y \Psi(x - y) \\ &= - \sum_{k=1}^N \alpha_k \psi_1(\|x - x_k\|) \langle x - x_k, f(x_k) \rangle. \end{aligned} \quad (2.9)$$

The α_k are unknown and are determined by solving the following equation for each $x_i \in X_N$

$$v'(x_i) = g(x_i). \quad (2.10)$$

This leads to the following system,

$$A\alpha = G,$$

where $\alpha \in \mathbb{R}^{N \times 1}$ and the i^{th} entry of α is α_i . $G \in \mathbb{R}^{N \times 1}$ and has the value $g(x_i)$ in its i^{th} entry. The collocation matrix $A \in \mathbb{R}^{N \times N}$ with the i^{th} , k^{th} entry of A , a_{ik} given by

$$\begin{aligned} a_{ik} &= (\delta_{x_i} \circ \mathcal{L})^x (\delta_{x_k} \circ \mathcal{L})^y \Psi(x - y) \\ &= -(\delta_{x_i} \circ \mathcal{L})^x \psi_1(\|x - x_k\|) \langle x - x_k, f(x_k) \rangle \\ &= -\psi_2(\|x_i - x_k\|) \langle x_i - x_k, f(x_i) \rangle \langle x_i - x_k, f(x_k) \rangle \\ &\quad - \psi_1(\|x_i - x_k\|) \langle f(x_i), f(x_k) \rangle. \end{aligned}$$

The linear system can be solved to give α . We can then construct the ansatz $v(x)$ relative to the grid points X_N . The approximant v and its orbital derivative are given by

$$\begin{aligned} v(x) &= - \sum_{k=1}^N \alpha_k \psi_1(\|x - x_k\|) \langle x - x_k, f(x_k) \rangle \\ v'(x) &= - \sum_{k=1}^N \alpha_k \left[\psi_2(\|x - x_k\|) \langle x - x_k, f(x) \rangle \langle x - x_k, f(x_k) \rangle \right. \\ &\quad \left. + \psi_1(\|x - x_k\|) \langle f(x), f(x_k) \rangle \right]. \end{aligned}$$

2.8 Error Estimates

We give a brief overview of the existing literature that shows that symmetric meshless collocation with radial basis functions for generalised linear partial differential equations is convergent with respect to the mesh norm also known as the fill distance. For a deeper analysis of this subject we point the reader towards [Giesl, 2007a] and [Giesl and Wendland, 2007], and for a deeper look at the construction of the functions spaces described see [Wendland, 2005]. Given a set K that contains grid points x_k the fill distance is the radius of the largest open ball that can fit into the set and does contain another grid point.

Definition 2.8.1. (*Fill Distance*). Let $K \subset \mathbb{R}^n$ be a compact set and let $X_N := \{x_1, \dots, x_N\} \subset K$ be a grid. The positive real number

$$h := h_{K, X_N} = \max_{y \in K} \min_{x \in X_N} \|x - y\|$$

is called the fill distance of X_N in K . In particular, for all $y \in K$ there is a grid point $x_k \in X_N$ such that $\|y - x_k\| \leq h$.

An important concept in the analysis of convergence using Wendland functions is the Native space \mathcal{F} , here we closely follow the presentation given in [Giesl, 2007a]. When Ψ is a Wendland function both \mathcal{F} and its dual \mathcal{F}^* are Sobolov spaces. \mathcal{F} is a function space that includes both the approximant q and the function it approximates Q . Where its dual \mathcal{F}^* is a space of operators including the linear operators used to construct the collocation matrix i.e. $(\delta_x \circ \mathcal{D})$. To see how native spaces are related to reproducing kernel Hilbert spaces see [Wendland, 2005]. Before we define the native space we informally introduce the Fourier transform and Schwartz space, a strict handling of these can be found in [Wendland, 2005]. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a space of rapidly decreasing smooth functions i.e. all of its derivatives decay faster than any polynomial, $\mathcal{S}'(\mathbb{R}^n)$ denotes its dual, i.e. the space of continuous linear operators on $\mathcal{S}(\mathbb{R}^n)$. For distributions $\lambda \in \mathcal{S}'(\mathbb{R}^n)$ we define as usual $\langle \check{\lambda}, \varphi \rangle := \langle \lambda, \check{\varphi} \rangle$ and $\langle \hat{\lambda}, \varphi \rangle := \langle \lambda, \hat{\varphi} \rangle$ with $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $\check{\varphi}(x) = \varphi(-x)$ and $\hat{\varphi}(w) = \int_{\mathbb{R}^n} \varphi(x) e^{-ix^T w} dx$ denotes the Fourier transform. The space $\mathcal{E}'(\mathbb{R}^n)$ denotes the distributions with compact support. As Ψ is a Wendland function we have $\Psi \in \mathcal{E}'(\mathbb{R}^n)$ as it has compact support. The Fourier transform $\hat{\Psi}(w) =: \varphi(w)$ is an analytic function and $\varphi(w) > 0$ holds for all $w \in \mathbb{R}^n$, cf. Proposition 3.11 in [Giesl, 2007a]. With this information we can now define the native space.

Definition 2.8.2. *Let $\Psi \in \mathcal{E}'(\mathbb{R}^n)$ be defined by a Wendland function and denote its Fourier transform by $\varphi(w) := \hat{\Psi}(w)$. We define the Hilbert space*

$$\mathcal{F}^* := \left\{ \lambda \in \mathcal{S}'(\mathbb{R}^n) \mid \hat{\lambda}(w)(\varphi(w))^{\frac{1}{2}} \in L_2(\mathbb{R}^n) \right\},$$

with the scalar product

$$\langle \lambda, \mu \rangle_{\mathcal{F}^*} := (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\lambda}(w) \overline{\hat{\mu}(w)} \varphi(w) dw.$$

*The native space \mathcal{F} of approximating functions is identified with the dual \mathcal{F}^{**} of \mathcal{F}^* . The norm is given by $\|f\|_{\mathcal{F}} := \sup_{\lambda \in \mathcal{F}^*, \lambda \neq 0} \frac{|\lambda(f)|}{\|\lambda\|_{\mathcal{F}^*}}$.*

Native spaces are useful as they allow the approximation error to be split between \mathcal{F} and \mathcal{F}^* , furthermore the approximant is norm-minimal with respect to the native space norm so we get that $\|Q - q\|_{\mathcal{F}} \leq \|Q\|_{\mathcal{F}}$ cf. Proposition 3.34 of [Giesl, 2007a]. In the convergence proof of Chapter 4 we make use of the native space. We give an example of the native space

with respect to that approximation. Take the dynamical system $\dot{x} = f(x)$, with $x \in \mathbb{R}^2$ and $f \in C^\sigma(\mathbb{R}^2, \mathbb{R}^2)$, with $\sigma \geq 2$. Let $X_N = \{x_1, \dots, x_N\}$ be a set of pairwise distinct points in a compact set K with $K \subset \mathbb{R}^2$. Let the differential operator \mathcal{D} be such that $\mathcal{D}q = q''\|f\| - q'\|f\|'$ where $\|\cdot\|$ is the Euclidean norm. Consider finding an approximation to the function Q that solves the equation $\mathcal{D}Q(x) = L(x)\|f(x)\|' - L'(x)\|f(x)\|$ where L is as defined in Theorem 1.2.1. Let Ψ be a sufficiently smooth Wendland function. Then we can define the native space \mathcal{F}_{X_N} and $\mathcal{F}_{X_N}^*$, that the approximant q of Q lives in, by

$$\mathcal{F}_{X_N}^* := \left\{ \lambda \in \mathcal{S}'(\mathbb{R}^2) \mid \lambda = \sum_{k=1}^N \alpha_k (\delta_{x_k} \circ \mathcal{D}), \alpha_k \in \mathbb{R} \right\},$$

$$\mathcal{F}_{X_N} := \{ \lambda * \Psi \mid \lambda \in \mathcal{F}_{X_N}^* \},$$

with $F_{X_N}^* \subseteq F^*$ and $F_{X_N} \subseteq F$. In [Giesl and Wendland, 2007] a stronger convergence result is given than the one in [Giesl, 2007a]. This is the convergence result we make use of in this work. We do not go into detail about how it is derived but we give a brief overview. One of the key results that it uses is the scattered zeros result from [Narcowich et al., 2005]. This states that if a function $Q \in W_p^\tau(\Omega)$, where Ω is a bounded set $\Omega \subset \mathbb{R}^n$, has a smooth enough boundary and is zero on a dense enough set of data points X then $|Q|_{W_q^m(\Omega)}$ will be bounded by $Ch^a|Q|_{W_p^\tau(\Omega)}$ where a is dependent on the smoothness of Q, p, q and \mathcal{D} . As we have used collocation we have zeros for $\mathcal{D}Q - \mathcal{D}q$ at all of the data points in the grid. Hence a result of the form $\|\mathcal{D}Q - \mathcal{D}q\|_{W_p^m(\Omega)} \leq Ch^a \|\mathcal{D}Q - \mathcal{D}q\|_{W_q^\tau(\Omega)}$ can be derived. The paper then goes on to show that the L_2 norm of a function with a linear differential operator can be bounded by a Sobolov norm of the function i.e. $\|\mathcal{D}Q\|_{L_2(\Omega)} \leq C\|Q\|_{W_2^m(\Omega)}$. The paper then extends the function Q to \mathbb{R}^n in a way such that it coincides with the original function on Ω and is in $W_2^\tau(\mathbb{R}^n)$. The authors then uses the equivalence of the norm induced by Ψ with the Sobolov norm $W_2^\tau(\mathbb{R}^n)$ and the fact that the approximant is norm-minimal in its induced norm to prove the convergence. This comes together in Theorem 3.5 of [Giesl and Wendland, 2007], in the theorem $\lfloor \cdot \rfloor$ denotes the standard floor function, $\lfloor x \rfloor := \max\{m \in \mathbb{Z} \mid m \leq x\}$.

Theorem 2.8.1. *Suppose Ψ is a reproducing kernel of $W_2^\tau(\mathbb{R}^n)$ with $k := \lfloor \tau \rfloor > m + n/2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having a Lipschitz boundary. Let \mathcal{D} be a linear differential operator of order m with coefficients $c_\alpha \in W_\infty^{k-m+1}(\Omega)$. Finally let q be the generalised*

interpolant to $Q \in W_2^\tau(\Omega)$ built in the manner described in this chapter. If $X \subseteq \Omega$ has a sufficiently small fill distance h_X , then for $1 \leq p \leq \infty$ the error estimate

$$\|\mathcal{D}Q - \mathcal{D}q\|_{L_p(\Omega)} \leq Ch_X^{\tau-m-n(1/2-1/p)_+} \|u\|_{W_2^\tau(\Omega)}, \quad (2.11)$$

is satisfied.

A version of this theorem, adapted for a boundary value problem, is used in Chapter 3. This result shows that our generalised interpolants will converge with respect to the differential operator. In both Chapters 3 and 4 the bound that this result delivers is not the bound we are looking for. In both cases we integrate along solutions to obtain the bounds we are looking for. This is more complicated in Chapter 4 where we do not have a boundary condition. We end up using the periodic orbit as a surrogate for the boundary condition.

2.9 Literature Review of Meshless Collocation and Dynamical Systems

We give an overview of existing work that applies meshfree collocation to dynamical systems. For a broader study on the topic of computational methods for dynamical systems see [Giesl and Hafstein, 2015]. In the early development of symmetric meshfree collocation all of the error estimates were for linear partial differential equations with constant coefficients such as [Franke and Schaback, 1998]. When the differential operator is the orbital derivative the coefficients will be of the form $f_i(x)$ where f is from (2.8). In [Giesl and Wendland, 2007] error estimates for partial differential equations with non-constant coefficients were given. That paper also goes on to apply meshless collocation to approximate a Lyapunov function for dynamical systems. This is achieved by using radial basis functions to numerically solve a modified version of Zubov's equation, $V'(x) = -\|x - x_0\|^2$. A more in depth look at using meshless collocation to approximate global Lyapunov functions is given in the book [Giesl, 2007a]. The study of using radial basis functions to approximate Lyapunov functions was further extended in [Giesl, 2007c], where Lyapunov functions for discrete dynamical systems are approximated. By determining level sets of the approximant to the Lyapunov function one is able determine a subset of the basin of attraction

for an equilibrium point. However the method is not able to comment on the attractivity of a small neighbourhood of the solution, where the orbital derivative becomes positive. This problem is addressed in [Giesl, 2008], where the author takes a Taylor approximation $n(x)$ of the Lyapunov function $V(x)$ and uses radial basis functions to approximate a function $W(x) := \frac{V(x)}{n(x)}$ that satisfies the equations $n'(x)W(x) + n(x)W'(x) = -\|x - x_0\|$ and $W(x_0) = 1$ by w . It is shown that w' is negative in the local neighbourhood of x_0 and hence w can be used to construct true subsets of the basin of attraction of x_0 . In [Giesl and Wendland, 2009] nonautonomous time-periodic RBFs are considered, where the flow is described by a function of the form $f(t + T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. The Lyapunov function is approximated and used to give a subset of the basin of attraction. In [Giesl and Wendland, 2012] the authors consider asymptotically autonomous ordinary differential equations. In [Giesl, 2012] the approximation of Lyapunov functions on finite-time domains is considered, and a method to determine a large subset of the domain of attraction for a solution is given.

Radial basis functions have also been used to approximate local contraction functions to determine attractive sets of dynamical systems. These are similar methods to those used in Chapter 3 and Chapter 4. In [Giesl, 2007b] radial basis functions are used to approximate a generalised Borg function that is then used to determine the basin of attraction of a periodic orbit in \mathbb{R}^2 . This work is then extended in [Giesl, 2009] where the basin of attraction of a periodic orbit is determined in three or more dimensions. A generalised Borg function is used to determine the basin of attraction in a neighbourhood of the periodic orbit. Outside that neighbourhood a Lyapunov function is used to determine the basin of attraction. In [Mohammed and Giesl, 2015] an adaptive method for grid refinement is studied. In [Bouvrie and Hamzi, 2011] reproducing kernel Hilbert spaces (RKHS) are used to construct a control. In [Rezaiee Pajand and Moghaddasie, 2012] a Lyapunov function for nonlinear systems is approximated by meshless collocation.

Chapter 3

Approximating the area of exponential attraction of finite-time dynamical systems in \mathbb{R}

3.1 Introduction

In this chapter we introduce a new method for approximating the area of exponential attraction for a finite-time dynamical system. Finite-time dynamical systems are a relatively new area of study, of which we give an overview of the current state of research in Section 1.2.1. The concept of finite-time areas of exponential attraction was introduced in [Giesl and Rasmussen, 2012], there the authors introduced a concept of attraction that allows for the distance between trajectories to increase for parts of their evolution as long as that distance increase is made up so that the overall distance decreases. They go on to show that if a finite-time Borg criterion is bounded then it is both a sufficient and necessary condition for a set to be an area of exponential attraction. We show that when the spatial dimension is limited to one then the finite-time Borg Criterion can be characterised by the orbital derivative of a scalar weight function W . We go on to show that the necessity Theorem from [Giesl and Rasmussen, 2012] can be reworked to give a partial differential equation involving W and an unknown term γ which is a given solution's rate of exponential attraction. By using the fact that γ is constant along solutions we show that by taking the orbital derivative one can derive a differential equation where W is the only unknown. We then go on to show how W can be approximated using meshfree collocation, and that

once W is known the area of exponential attraction can be approximated. We show that the method we introduce in this chapter to determine the area of exponential attraction is convergent. Finally we apply the method to two examples.

Section 3.2 covers the theoretical foundations. It presents some definitions for finite-time sets and introduces some concepts of attraction in finite-time. It goes on to give definitions of an area of exponential attraction and then introduces the sufficiency result from [Giesl and Rasmussen, 2012]. Finally it shows the modified version of the necessity result delivers a differential equation containing W . In Section 3.3 we give some corollaries to show that a linear differential equation where W is only unknown can be formed. We then show that level sets of W' can characterise an area of exponential attraction. We then describe how the boundary value problem for W can be approximated using radial basis functions, giving all of the components needed for others to replicate the procedure. In Section 3.3.2 we show that the collocation method described in Section 3.3 is convergent with respect to a mesh-norm. Finally in Section 3.4 we give two examples of the method being applied, one an analytically solvable problem and the other which can only be solved numerically.

3.2 Theoretical Foundations

In this section we give some of the theoretical foundations concerning attraction in finite-time dynamical systems. We start by presenting some pre-existing work that is required to help present our later work. Firstly we introduce some finite-time definitions of some set properties. We then present some finite-time definitions of attraction, a definition for the domain of attraction of a solution and finally go on to define the area of exponential attraction. These definitions were originally given in [Giesl and Rasmussen, 2012]. We then go on to give a definition of a finite-time Borg-criterion and give the sufficiency result from [Giesl and Rasmussen, 2012], which shows that the finite-time Borg-criterion is a sufficient condition for a connected set to be an area of exponential attraction. We also present a new corollary that shows that by limiting the spatial dimension to one we can characterise the finite-time Borg-criterion by a scalar weight function rather the Riemannian metric given in its original definition. Finally we give a modified version of the necessity theorem from [Giesl and Rasmussen, 2012], we modify it so that it gives a partial differential equation for W , which is a new result.

3.2.1 Preliminaries

We are interested in the nonautonomous differential equations of the form

$$\dot{x}(t) = f(t, x), \quad (3.1)$$

with $x \in \mathbb{R}^n$, $t \in [T_1, T_2] := \mathbb{I}$, with $\infty < T_1 < T_2 < \infty$ and $f : \mathbb{I} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f \in C^1$, and assume that solutions to IVPs exist for all $t \in \mathbb{I}$ and are unique (see [Chicone, 2000] Theorem 1.2 for a local existence result). We define a general solution as $\varphi : \mathbb{I} \times \mathbb{I} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\varphi(t, t_0, x_0)$ being the solution of the initial value problem of the system (3.1) with the starting position given by $x(t_0) = x_0$, and the first variable being the time at which the solution's current position is given. At times a shorter notation for a solution may be used: let $\mu : \mathbb{I} \rightarrow \mathbb{R}^n$ be the solution with $\mu(T_1) = \xi$, hence $\mu(t) = \varphi(t, T_1, \xi)$. We introduce the following definitions from [Giesl and Rasmussen, 2012], these three definitions help define properties of the sets that we will use later in the chapter.

Definition 3.2.1. (*t-fibre*). Let $G \subset \mathbb{I} \times \mathbb{R}^n$. A *t-fibre* of G is the maximal subset of the phase space contained in G for a given time t , i.e.

$$G(t) := \{x \in \mathbb{R}^n : (t, x) \in G\}.$$

The t-fibre allows one to isolate the spacial component of the phase space at a given time, for example $G(T_1) \cup G(T_2)$ represents the spatial points that are on the temporal boundary of G . A nonautonomous set will be one that has a non-empty t-fibre for every time $t \in \mathbb{I}$.

Definition 3.2.2. (*Nonautonomous set*). A set G with $G \subset \mathbb{I} \times \mathbb{R}^n$ is called *nonautonomous* if the t-fibre $G(t) \neq \emptyset$, for all $t \in \mathbb{I}$, furthermore G is called *connected*, *compact* and *open* if all the t-fibres are respectively *connected*, *compact* and *open*.

Definition 3.2.3. (*Invariant set*). A nonautonomous set G with $G \subset \mathbb{I} \times \mathbb{R}^n$ is called *positively invariant*, with respect to (3.1), if $\varphi(t, \tau, G(\tau)) \subset G(t)$ and *invariant* if $\varphi(t, \tau, G(\tau)) = G(t)$, for all $t > \tau$ and $t, \tau \in \mathbb{I}$.

Finally we give a definition of the distance between a set and a point.

Definition 3.2.4. (*Distance between a set and a point*). Let q be a point such that $q \in \mathbb{R}^n$

then its distance from a set $S \subseteq \mathbb{R}^n$ is defined as

$$\text{dist}(q, S) = \inf_{x \in S} \|q - x\|.$$

The following definitions come from [Giesl and Rasmussen, 2012], in that paper the authors were interested in introducing more flexible concepts of attraction for finite-time dynamical systems. In [Haller, 2000] a stronger concept of finite-time attraction was introduced where solutions are required to reduce the Euclidean distance between nearby trajectories at all points in space-time. In [Giesl and Rasmussen, 2012] the weaker concept of attraction requires the distance between trajectories with respect to a norm at the ending time, T_2 , to be less than the distance between trajectories at the starting time, T_1 . This definition allows the distance between nearby trajectories to increase for part of their evolution along the time interval as long as they make up for that by reducing the distance at other times along the time interval. The difference between the two conditions is similar to the difference between the original Borg's Criterion (Theorem 1.2.1) and the Riemannian metric Borg criterion (Theorem 4.2.1). The original Borg criterion required the Euclidean distance between trajectories to decrease at all times whereas the Riemannian version allows for the Euclidean distance between trajectories to move away from each other along parts of their orbit as long as the distance is made up at other times along the trajectories. It should be noted that the definition of a solution being finite-time attractive is dependent on the choice of norm.

Definition 3.2.5. (*Finite-time attractivity*).

1. Let $\mu : \mathbb{I} \rightarrow \mathbb{R}^n$ be the solution of (3.1) then a solution μ is attractive on \mathbb{I} if there exists $\eta > 0$ such that

$$\|\varphi(T_2, T_1, x_0) - \mu(T_2)\| < \|\varphi(T_2, T_1, x_0) - \mu(T_1)\| \quad \text{for all } x \in B_\eta(\mu(T_1)) \setminus \{\mu(T_1)\}.$$

2. A solution is μ exponentially attractive on \mathbb{I} if

$$\limsup_{\eta \searrow 0} \frac{1}{\eta} \text{dist}(\varphi(T_2, T_1, B_\eta(\mu(T_1))), \mu(T_2)) < 1,$$

3. and the negative number

$$\frac{1}{T} \log \left(\limsup_{\eta \searrow 0} \frac{1}{\eta} \text{dist}(\varphi(T_2, T_1, B_\eta(\mu(T_1))), \mu(T_2)) \right)$$

is the rate of exponential attraction.

We now introduce two definitions of attractive sets for finite-time dynamical systems, both were originally given in [Giesl and Rasmussen, 2012]. The first is for the domain of attraction of a solution μ . The domain of attraction of a solution of a finite-time dynamical system is a connected and invariant nonautonomous set that contains the graph μ and solutions that are attracted to μ in the manner given by Definition 3.2.5. The second is an area of exponential attraction, we will later present a method to approximate this area. The area of exponential attraction differs from traditional concepts of asymptotic areas of attraction as it is not focused on a single solution. This is because in the classical case of asymptotic attraction we are usually interested in an invariant set that all other solutions will move towards. However finite-time systems lack sufficient time for the long-term behaviour to emerge and as such individual solutions do not play a dominant role in their study. Instead the area of exponential attraction focuses on solutions that are exponentially attracted to each other on the time interval.

Definition 3.2.6. (*Domain of attraction for a solution*). Let $\mu : \mathbb{I} \rightarrow \mathbb{R}^n$ be an attractive solution on \mathbb{I} . Then a connected and invariant nonautonomous set $G_\mu \subset \mathbb{I} \times \mathbb{R}^n$ is called a domain of attraction of μ if

$$\|\varphi(T_2, T_1, x) - \mu(T_2)\| < \|x - \mu(T_1)\| \text{ for all } x \in G_\mu(T_1) \setminus \{\mu(T_1)\},$$

G_μ is the maximal set such that this is true (with respect to set inclusion) and which contains the graph of μ .

Definition 3.2.7. (*Areas of attractivity*). A connected and invariant nonautonomous set $G \subset \mathbb{I} \times \mathbb{R}^n$ is called:

1. An area of attraction if all solutions in G are attractive.
2. An area of exponential attraction if all solutions in G are exponentially attractive.

Remark 3.3 of [Giesl and Rasmussen, 2012] states that the area of attraction can be characterised by the initial T-fibre of G , (i.e. $G(T_1)$). The following is Proposition 2.5 from [Giesl and Rasmussen, 2012], here it is restricted to the case where the phase-space is one-dimensional, i.e. $n = 1$. The proposition will be used in the proof of Theorem 3.2.2.

Proposition 3.2.1. *Denote by $F_{T_2} : \mathbb{R} \rightarrow \mathbb{R}$ the time-map of (3.1), which is defined by $F_{T_2}(x) := \varphi(T_2, T_1, x)$, moreover let $\mu : \mathbb{I} \rightarrow \mathbb{R}$ be a solution of (3.1). Then the following statements hold true:*

1. *if μ is attractive on \mathbb{I} , then $\left(\frac{\partial}{\partial x} F_{T_2}(\mu(T_1))\right)^2 \leq 1$,*
2. *μ is exponentially attractive on \mathbb{I} if and only if $\left(\frac{\partial}{\partial x} F_{T_2}(\mu(T_1))\right)^2 < 1$. The rate of exponential attraction, ν on \mathbb{I} is given by*

$$\nu := \frac{1}{2(T_2 - T_1)} \log \left(\left(\frac{\partial}{\partial x} F_{T_2}(\mu(T_1)) \right)^2 \right).$$

From this we can see that as long as $f \in C^1$ then ν will be continuous with respect to x as it is characterised by the solution at the end time, T_2 . In the case of systems in higher spatial dimensions the rate of exponential attraction is dependent on the eigenvalues of $(DF_{T_2}(t, x))^T (DF_{T_2}(t, x))$. The proof is not included here but can be found in [Giesl and Rasmussen, 2012].

In the following section we introduce the sufficiency theorem from [Giesl and Rasmussen, 2012], the theorem shows that if a finite-time Borg criterion is satisfied on a connected and invariant nonautonomous set it is a sufficient condition for the set to be an area of exponential attraction with a minimal rate of attraction $-\nu$. We start by defining a time-varying Riemannian metric and use it to define the function L_M which is our finite-time-Borg function, we then show a simplified version of L_M which only requires a scalar weight function W rather than the time-varying Riemannian M , before giving the sufficiency result. We start with some definitions.

Definition 3.2.8. *(Time-varying Riemannian metric). A continuously differentiable function $M : \mathbb{I} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is a time-varying Riemannian metric if $M(t, x)$ is symmetric and positive definite for all $(t, x) \in \mathbb{I} \times \mathbb{R}^n$.*

As we are interested in the case of $n = 1$ our time-varying-Riemannian metric will be positive scalar. In [Giesl and Rasmussen, 2012] the following definition is given for the Borg-like function.

Definition 3.2.9. (L_M). *Given a Riemannian metric M we define the following function*

$$L_M(t, x; v) := v^T \left(M(t, x) D_x f(t, x) + \frac{1}{2} M'(t, x) \right) v,$$

and

$$L_M(t, x) := \max_{v \in \mathbb{R}^n, v^T M(t, x) v = 1} L_M(t, x; v).$$

Where $M'(t, x) := M_t + \langle \nabla M(t, x), f(t, x) \rangle$.

By restricting the spatial dimension to one we can take the Riemannian metric to be of the form $e^{2W(t, x)}$ where $W \in C^1(\mathbb{I} \times \mathbb{R}, \mathbb{R})$, this is shown in the proceeding corollary. It is a similar result to that given by Corollary 14 from [Giesl, 2004a] for the periodic orbit case.

Corollary 3.2.1. *If the restrictions that $n = 1$ and $M(t, x) = e^{2W(t, x)}$ are imposed with $W \in C^1(\mathbb{I} \times \mathbb{R}, \mathbb{R})$ then $L_M(t, x)$ satisfies the following*

$$L_M(t, x) = f_x(t, x) + W'(t, x).$$

Proof. Firstly note that with $n = 1$ the maximum condition in the definition of L_M becomes $M(t, x) = \frac{1}{v^2}$ with $v \in \mathbb{R}$, so $L_M(t, x) = f_x(t, x) + \frac{M'(t, x)}{2M(t, x)}$, then the result follows. \square

Here we introduce the sufficiency condition (Theorem 4.2) from [Giesl and Rasmussen, 2012], as all connected intervals of \mathbb{R} will be convex we can also absorb Corollary 4.3 from [Giesl and Rasmussen, 2012] into the theorem. We give a version that uses the result of Corollary 3.2.1 to reduce the Riemannian metric to a scalar weight function.

Theorem 3.2.1. *Consider the differential equation (3.1), let $G \subset \mathbb{I} \times \mathbb{R}$ be a nonempty, connected, compact and invariant nonautonomous set and let $W \in C^1(G)$ with $W(T_1, x) = 0$ for all $x \in G(T_1)$ and $W(T_2, x) = 0$ for all $x \in G(T_2)$. Let $M(t, x) = e^{2W(t, x)}$ and assume*

that there exists a $\nu > 0$ such that

$$L_M(t, x) \leq -\nu, \quad \text{for all } (t, x) \in G.$$

Then G is an area of exponential attraction. Furthermore

$$\|\varphi(T_2, T_1, x) - \varphi(T_2, T_1, y)\| \leq e^{-\nu(T_2 - T_1)} \|x - y\| \quad \text{for all } x, y \in G(T_1),$$

i.e. all solutions in G are exponentially attractive with a rate of at least ν .

Although this is not new work we include the proof as it is instructive in understanding how the finite-time-Borg criterion relates to areas of exponential attraction. The proof works by showing that if the finite-time-Borg function is bounded above by a negative number then the distance, with respect to the weight-function, between the trajectories within a δ -neighbourhood of any point in the t -fibre $G(T_1)$ will decrease at an exponential rate of at least ν at time T_2 . Finally the result is expanded beyond the δ -neighbourhood by using that the t -fibre $G(T_1)$ is connected and hence convex as it is in \mathbb{R} .

Proof. We start by showing that for all $\gamma < \nu$, there exists a $\delta > 0$ such that for all $x_0 \in G(T_1)$, we have the following

$$\|\varphi(T_2, T_1, x_0) - \varphi(T_2, T_1, \eta)\| \leq e^{-\nu(T_2 - T_1)} \|x_0 - \eta\| \quad \text{for all } \eta \in B_\delta(x_0). \quad (3.2)$$

Let $\gamma < \nu$, as f_x (the derivative of f with respect to x) is continuous it is uniformly continuous on G , hence there exists a $\delta_1 > 0$ such that

$$\|f_x(t, x_0) - f_x(t, \eta)\| \leq \nu - \gamma \quad \text{for all } (t, x_0) \in G \text{ and } \eta \in B_{\delta_1}(x_0).$$

Fix $x_0 \in G(T_1)$ and $\eta \in \mathbb{R}$ with $\|\eta - x_0\| \leq \delta := \frac{\delta_1}{2}$ and let $\mu(t) = \varphi(t, T_1, x_0)$ for all $t \in \mathbb{I}$.

We define a time-varying distance by

$$\Gamma(t) := e^{W(t, \mu(t))} \|\varphi(t, T_1, \eta) - \mu(t)\| \quad \text{for all } t \in \mathbb{I}. \quad (3.3)$$

We note that $\Gamma(T_1) = \|\eta - x_0\|$ giving that $\Gamma(T_1) \leq \frac{\delta_1}{2}$ and hence there exists a maximal $\theta > T_1$ such that $\Gamma(t) \leq \delta_1$ for all $t \in [T_1, \theta]$. To show that γ decreases exponentially on

$[T_1, \theta]$ we take the temporal derivative of Γ^2

$$\begin{aligned}
\frac{d}{dt}\Gamma^2(t) &= 2W'(t, \mu(t))e^{2W(t, \mu(t))}\|\varphi(t, T_1, \eta) - \mu(t)\|^2 \\
&\quad + 2e^{2W(t, \mu(t))}(\varphi(t, T_1, \eta) - \mu(t))(f(t, \varphi(t, T_1, \eta)) - f(t, \mu(t))), \\
&= 2\Gamma^2(t)W'(t, \mu(t)) + 2\Gamma^2(t) \int_0^1 f_x(t, \mu(t) + \lambda(\varphi(t, T_1, \eta) - \mu(t)))d\lambda, \\
&= 2\Gamma^2(t) \underbrace{\left(W'(t, \mu(t)) + f_x(t, \mu(t))\right)}_{=L_M(t, \mu(t)) \leq -\nu} \\
&\quad + 2\Gamma^2(t) \int_0^1 \left[f_x(t, \mu(t) + \lambda(\varphi(t, T_1, \eta) - \mu(t))) - f_x(t, \mu(t))\right]d\lambda, \\
&\leq 2(\nu - \gamma - \nu)\Gamma^2(t).
\end{aligned}$$

Hence

$$\Gamma(t) \leq e^{-\gamma(t-T_1)}\Gamma(T_1) \text{ for all } t \in [T_1, \theta]. \quad (3.4)$$

Therefore $\Gamma(t) \leq \frac{\delta_1}{2}$ for all $t \in [T_1, \theta]$, if $\theta < T_2$ then the maximality of θ is contradicted, hence it holds for all $t \in [T_1, T_2]$. As a result (3.2) holds true for any $x_0 \in G(T_1)$.

To extend the proof over all $x, y \in G(T_1)$ we follow the methodology of Corollary 4.3 from [Giesl and Rasmussen, 2012]. Fix x and y as arbitrary values in $G(T_1)$. We note that there exists a $\kappa \in \mathbb{R}^+$ with $\kappa < \delta$ and $m \in \mathbb{N}$ such that $\kappa m = \|x - y\|$. Therefore $y = x + \kappa m \frac{y - x}{\|y - x\|}$. Since $G(T_1)$ is connected and we are working in \mathbb{R} it is convex, hence we can take $x_i = x + \kappa i \frac{y - x}{\|y - x\|}$ for $i = 0, \dots, m$ and will have that each $x_i \in G(T_1)$ for $i = 0, \dots, m$. Then for each x_i and x_{i+1} such that $i < m$ we can make use of (3.4). Through repeated use of the above argument we get the desired result that

$$\|\varphi(T_1, T_2, x) - \varphi(T_1, T_2, y)\| \leq e^{-\gamma(T_2-T_1)}\|x - y\|.$$

□

The next theorem is based upon the necessity results, Theorem 5.1, from [Giesl and Rasmussen, 2012]. However it is a modified version. Firstly we use the weight function version of L_M introduced in Corollary 3.2.1. Secondly it gives a partial differential equation containing W , Theorem 5.1 in [Giesl and Rasmussen, 2012] only gives an inequality containing

the finite-time Riemannian metric. This is a new result and will later allow for a collocation equation to be formed that will be used to approximate W' and hence the area of exponential attraction.

Theorem 3.2.2. *Consider the differential equation (3.1) with $n = 1$ and $f \in C^\sigma(\mathbb{I} \times \mathbb{R}, \mathbb{R})$ with $\sigma \geq 2$ and a compact nonautonomous set $G \subset \mathbb{I} \times \mathbb{R}$ which is an area of exponential attraction. Let $-\nu < 0$ be the maximal rate of exponential attraction of all solutions in G . Then there exists a continuous function $W : G \rightarrow \mathbb{R}$ with $W \in C^{\sigma-1}(G)$ and $W(T_1, x) = W(T_2, x) = 0$ for $x \in \mathbb{R}$ such that*

$$L_M(t, x) = f_x(t, x) + W'(t, x) \leq -\nu \text{ for all } (t, x) \in G,$$

and

$$f_x(t, \varphi(t, T_1, x)) + W'(t, \varphi(t, T_1, x)) = \gamma(x) \text{ for all } (t, x) \in \mathbb{I} \times G(T_1). \quad (3.5)$$

Where $\gamma(x)$ is the exponential rate of attraction of the solution $\varphi(t, T_1, x)$ on \mathbb{I} with initial condition x .

Proof. We start by considering a fixed initial value and its associated solution, we begin to construct a Riemannian metric M for that solution in similar fashion as Theorem 5.1 in [Giesl and Rasmussen, 2012]. Fix a $x_0 \in G(T_1)$ and let μ be the associated solution with $\mu(t) = \varphi(t, T_1, x_0)$. Consider the variational form associated with this solution

$$\dot{y} = f_x(t, \mu(t))y. \quad (3.6)$$

We follow an approach based on Floquet theory, denote the (fundamental) solution of (3.6) as $\phi : \mathbb{I} \rightarrow \mathbb{R}$, then the solution will take the form $\phi(t) = e^{\int_{T_1}^t f_x(\tau, \mu(\tau))d\tau}$. We build the metric in a similar way to how $P(t)$ from Floquet theory is built (Section 2.4 [Chicone, 2000]) and use the fact that the solution of the variational form satisfies $\phi(t) = \frac{\partial}{\partial x} F_t(\mu(T_1))$, where F_t is the time map defined in Proposition 3.2.1. We combine this along with Proposition 3.2.1 to give

$$\gamma(x_0) = \frac{1}{2(T_2 - T_1)} \log(\phi^2(T_2)).$$

or conversely that

$$\phi(T_2) = e^{\gamma(x_0)(T_2-T_1)}.$$

Where $\gamma(x_0)$ is the rate of exponential attraction for the solution μ (from Proposition 3.2.1). It should be noted that due to its characterisation in Proposition 3.2.1 $\gamma(x)$ will be $C^{\sigma-1}$ in space and its orbital derivative satisfies $\gamma'(x) = 0$. The metric can be constructed as

$$M(t, \mu(t)) := \phi^{-2}(t) e^{2\gamma(x_0)(t-T_1)}. \quad (3.7)$$

We now define W as

$$W(t, x) := \frac{1}{2} \log(M(t, x)). \quad (3.8)$$

Therefore by combining (3.7) and (3.8) we get

$$\begin{aligned} W(t, \mu(t)) &= -\log(\phi(t)) + \gamma(x_0)(t - T_1), \\ &= -\int_{T_1}^t f_x(\tau, \mu(\tau)) d\tau + \gamma(x_0)(t - T_1). \end{aligned} \quad (3.9)$$

from this it is clear that $W \in C^{\sigma-1}$. We can take the orbital derivative to give

$$W'(t, \mu(t)) = -f_x(t, \mu(t)) + \gamma(x_0).$$

Since $M(T_1, \mu(T_1)) = M(T_2, \mu(T_2)) = 1$ then recalling from (3.8) that $M(t, x) = e^{2W(t, x)}$ we see that $W(T_1, \mu(T_1)) = W(T_2, \mu(T_2)) = 0$. Finally by virtue of G being an area of exponential attraction with maximal rate $-\nu$ and $L_M(t, x)$ we have the other result. \square

3.3 An Approximant for W

Theorem 3.2.2 gives an equation involving W' , however it cannot yet be used to construct an approximation to W' as γ is unknown. We seek to form a collocation equation that only involves a linear differential operation applied to W and known values on the right hand side. Proposition 3.2.1 shows γ is constant along solutions, this allows us to take the orbital derivative of (3.5) to derive a linear partial differential equation where W is

the only unknown. We also confirm that L_M is constant along solutions. These are both shown in the following corollary.

Corollary 3.3.1. *Let the conditions of Theorem 3.2.2 be satisfied, then the following two statements are true*

$$\begin{aligned} W''(t, x) &= -f_x(t, x)' \\ &= -(f_{xt}(t, x) + f_{xx}(t, x)f(t, x)) \quad \text{for all } (t, x) \in G, \end{aligned} \quad (3.10)$$

and

$$L'_M(t, x) = 0 \quad \text{for all } (t, x) \in G. \quad (3.11)$$

Proof. The proof of both statements is very simple, firstly we prove (3.10). From Theorem 3.2.2 we have that

$$f_x(t, \varphi(t, T_1, x)) + W'(t, \varphi(t, T_1, x)) = \gamma(x) \quad \text{for all } (t, x) \in \mathbb{I} \times G(T_1).$$

Then by taking the orbital derivative and noting that Proposition 3.2.1 gives that γ is constant along solutions we get (3.10). For the second statement we note that by Corollary 3.2.1 we have for all $(t, x) \in \mathbb{I} \times G(T_1)$,

$$L_M(t, x) = f_x(t, x) + W'(t, x).$$

Then by taking the orbital derivative of the above equation we get,

$$\begin{aligned} L'_M(t, x) &= (W'(t, x) + f_x(t, x))', \\ &= W''(t, x) + (f_{xt}(t, x) + f_{xx}(t, x)f(t, x)). \end{aligned}$$

Then by using (3.10) we immediately see (3.11) is true. \square

Corollary 3.3.1 gives a partial differential equation which can be numerically solved to give W . We can then use W to construct an approximation of the area of exponential attraction. To approximate a subset of the area of attraction with a given maximal rate of exponential attraction λ , we wish to find a level set of $L_M = \lambda$ which is equal to that

value. To clarify that this is an area of exponential attraction we bring together the results of Theorem 3.2.1 and Theorem 3.2.2 to explicitly state that the set bounded by the level sets where the rate of exponential attraction, λ , is an area of exponential attraction. The following corollary confirms that the function W that satisfies $W'' = -f'_x$ on the interior of the domain and $W = 0$ on the boundary is the same function W from Theorem 3.2.2 and that it characterises an area of exponential attraction.

Corollary 3.3.2. *Consider the differential equation (3.1) with $n = 1$ and $f \in C^\sigma(\mathbb{I} \times \mathbb{R}, \mathbb{R})$ with $\sigma \geq 2$ and a compact nonautonomous set $G \subset \mathbb{I} \times \mathbb{R}$ which is an area of exponential attraction. Let $-\nu < 0$ be the maximal rate of exponential attraction of all solutions in G . Then there exists a function $W \in C^{\sigma-1}(G, \mathbb{R})$ which satisfies $W''(t, x) = -(f_x(t, x))'$. Conversely consider a nonempty, connected, compact and invariant nonautonomous set $G_1 \subset \mathbb{I} \times \mathbb{R}$, assume the function $W \in C^{\sigma-1}(G_1, \mathbb{R})$ solves the boundary value problem*

- $W(t, x) = 0$ for all $(t, x) \in G_1(T_1)$,
- $W(t, x) = 0$ for all $(t, x) \in G_1(T_2)$,
- $W''(t, x) = -(f_x(t, x))'$ for all $(t, x) \in G_1$.

Then W is the function described in Theorem 3.2.2 and $L_M(t, \mu(t))$ is constant for all $t \in \mathbb{I}$. Furthermore there exist $x_1, x_2 \in G_1(T_1)$ with $x_1 < x_2$, $\varphi(t, T_1, x_1) \in G_1$, $\varphi(t, T_1, x_2) \in G_1$ satisfying the following conditions and $\lambda < 0$,

- $L_M(T_1, x_1) = \lambda$,
- $L_M(T_1, x_2) = \lambda$,
- $L_M(T_1, x) \leq \lambda$, for all $x \in (x_1, x_2)$.

Then the set enclosed by (T_1, x_1) to (T_1, x_2) , $(T_2, \varphi(T_2, T_1, x_1))$ to $(T_2, \varphi(T_2, T_1, x_2))$ and the lines $\{(t, \varphi(t, T_1, x_1)) \mid t \in \mathbb{I}\}$ and $\{(t, \varphi(t, T_1, x_2)) \mid t \in \mathbb{I}\}$ will be an area of exponential attraction on \mathbb{I} with maximal rate of exponential attraction λ .

Proof. We fix $x_0 \in G_1(T_1)$ and take $\mu(t) = \varphi(t, T_1, x_0)$. By integrating $W''(t, x)$ along

$\mu(t)$ we obtain the following

$$\begin{aligned} \int_{T_1}^t W''(\tau, \mu(\tau)) d\tau &= - \int_{T_1}^t (f_x(\tau, \mu(\tau)))' d\tau, \\ (\Leftrightarrow) \quad W'(t, \mu(t)) &= -f_x(t, \mu(t)) + c. \end{aligned} \quad (3.12)$$

We integrate (3.12) again between the temporal boundaries

$$0 = \int_{T_1}^{T_2} W'(\tau, \mu(\tau)) d\tau = - \int_{T_1}^{T_2} f_x(\tau, \mu(\tau)) d\tau + c(T_2 - T_1).$$

This gives,

$$c = \frac{\int_{T_1}^{T_2} f_x(\tau, \mu(\tau)) d\tau}{T_2 - T_1}. \quad (3.13)$$

From (3.9) we have $0 = W(T_2, \mu(T_2)) = - \int_{T_1}^{T_2} f_x(\tau, \mu(\tau)) d\tau + \gamma(x_0)(T_2 - T_1)$. Rearranging this and (3.13) gives us that $c = \gamma(x_0)$. Therefore the function W given in this corollary satisfies $W'(t, \mu(t)) = -f_x(t, \mu(t)) + \gamma(x_0)$. It is therefore the same function as given in Theorem 3.2.2. Furthermore, as $L_M(t, \mu(t)) = \gamma(x_0)$ and $\gamma(x_0)$ is constant along solutions by definition cf. Proposition 3.2.1 we have that $L'_M(t, \mu(t)) = 0$. The second part of the corollary immediately follows from Theorem 3.2.2 where it is shown that $\gamma(x_0)$ is the rate of exponential attraction for the solution $\varphi(t, T_1, x_0)$. \square

3.3.1 Meshfree Collocation

In order to construct the approximation to W and hence L_M we will use meshfree collocation with compactly supported radial basis functions. We will be using the symmetric method of generalised interpolation that is described in Chapter 2 to approximate the boundary value problem that is described below.

Let us define the region of interest as $G \subset \mathbb{I} \times \mathbb{R}$ and let the temporal boundary of G be called $\Gamma := G(T_1) \cup G(T_2)$. The boundary value problem can be described as

$$\begin{aligned} W''(t, x) &= -(f_x(t, x))' \quad \text{for } (t, x) \in G, \\ W(t, x) &= 0 \quad \text{for all } (t, x) \in \Gamma. \end{aligned} \quad (3.14)$$

Let us define our collocation points, the set of interior points $X := \{\tilde{x}_i = (t_i, x_i) \in G$ for $i = 1, \dots, N\}$ and data points on the temporal boundary $X_\Gamma := \{\tilde{\eta}_j = (\tau_j, \eta_j) \in \Gamma$ for $j = 1, \dots, N_1\}$. The ansatz for the approximation will be

$$w(t, x) = \sum_{k=1}^N \alpha_k (\delta_{\tilde{x}_k} \circ \mathcal{L})^{\tilde{y}} \Psi(\tilde{x} - \tilde{y}) + \sum_{k=1}^{N_1} \beta_k (\delta_{\tilde{\eta}_k})^{\tilde{y}} \Psi(\tilde{x} - \tilde{y}). \quad (3.15)$$

Where the differential operator $\mathcal{L}w(t, x) = w''(t, x)$ and the coefficient vector $(\alpha, \beta)^T \in \mathbb{R}^{N+N_1}$ is determined by the following collocation system

$$\begin{aligned} (\delta_{\tilde{x}_j} \circ \mathcal{L})(w) &= (\delta_{\tilde{x}_j} \circ \mathcal{L})(W) = -f'_x(\tilde{x}_j), \\ (\delta_{\tilde{\eta}_j})(w) &= (\delta_{\tilde{\eta}_j})(W) = 0. \end{aligned}$$

The linear system becomes

$$\begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

where $A \in \mathbb{R}^{N \times N}$ with entries $A_{ik} = (\delta_{\tilde{x}_i} \circ \mathcal{L})^{\tilde{x}} (\delta_{\tilde{x}_k} \circ \mathcal{L})^{\tilde{y}} \Psi(\tilde{x} - \tilde{y})$,

where $C \in \mathbb{R}^{N \times N_1}$ with entries $C_{ik} = (\delta_{\tilde{x}_i} \circ \mathcal{L})^{\tilde{x}} (\delta_{\tilde{\eta}_k})^{\tilde{y}} \Psi(\tilde{x} - \tilde{y})$,

where $B \in \mathbb{R}^{N_1 \times N_1}$ with entries $B_{ik} = (\delta_{\tilde{\eta}_i})^{\tilde{x}} (\delta_{\tilde{\eta}_k})^{\tilde{y}} \Psi(\tilde{x} - \tilde{y})$,

where $a \in \mathbb{R}^N$ with entries $a_i = -f'_x(\tilde{x}_i)$,

where $b \in \mathbb{R}^{N_1}$ with entries $b_i = 0$,

and $(\alpha, \beta)^T$ is a \mathbb{R}^{N+N_1} coefficient vector to be determined.

As the radial basis function is a positive definite function and we have used symmetric collocation this system has a solution. The following shows the results of the pointwise and differential operators being applied to a general radial basis function. We remind the reader of the recursive notation introduced in Chapter 2 to define the different differential powers of ψ , $\psi_i = \frac{1}{r} \frac{\partial \psi_{i-1}(r)}{\partial r}$. The first is the boundary point operator, $(\delta_{\tilde{\eta}_i})$, being applied to interior points of the ansatz, (3.15). This and the following results come from [Giesl,

2012]

$$\begin{aligned}
(\delta_{\tilde{\eta}_i})^{\tilde{x}}(\delta_{\tilde{x}_k} \circ \mathcal{L})^{\tilde{y}}\Psi(\tilde{x} - \tilde{y}) = \\
\psi_2(\|\tilde{\eta}_i - \tilde{x}_k\|) \left[(\tau_i - t_k)^2 + 2(\tau_i - t_k)(\eta_i - x_k)f(\tilde{x}_k) + (\eta_i - x_k)^2 f^2(\tilde{x}_k) \right] \\
+ \psi_1(\|\tilde{\eta}_i - \tilde{x}_k\|) \left[1 + f^2(\tilde{x}_k) - (\eta_i - x_k) \left(f_x(\tilde{x}_k) f(\tilde{x}_k) + f_t(\tilde{x}_k) \right) \right].
\end{aligned}$$

Now we show the interior operator, $(\delta_{\tilde{x}_i} \circ \mathcal{L})$, applied to the boundary points of the ansatz,

$$\begin{aligned}
(\delta_{\tilde{x}_i} \circ \mathcal{L})^{\tilde{x}}(\delta_{\tilde{\eta}_k})^{\tilde{y}}\Psi(\tilde{x} - \tilde{y}) = \\
\psi_2(\|\tilde{x}_i - \tilde{\eta}_k\|) \left[(t_i - \tau_k)^2 + 2(t_i - \tau_k)(x_i - \eta_k)f(\tilde{x}_i) + (x_i - \eta_k)^2 f^2(\tilde{x}_i) \right] \\
+ \psi_1(\|\tilde{x}_i - \tilde{\eta}_k\|) \left[1 + f^2(\tilde{x}_i) + (x_i - \eta_k) \left(f_x(\tilde{x}_i)f(\tilde{x}_i) + f_t(\tilde{x}_i) \right) \right].
\end{aligned}$$

Now we present the interior operator, $(\delta_{\tilde{x}_i} \circ \mathcal{L})$, applied to the interior points of the ansatz,

$$\begin{aligned}
& (\delta_{\tilde{x}_i} \circ \mathcal{L})^{\tilde{x}} (\delta_{\tilde{x}_k} \circ \mathcal{L})^{\tilde{y}} \Psi(\tilde{x} - \tilde{y}) = \\
& \psi_4(\|x_i - x_k\|) \left((t_i - t_k)^2 \left((t_i - t_k) + (x_i - x_k)(f(\tilde{x}_k) + f(\tilde{x}_i)) \right)^2 \right. \\
& \quad + \left((x_i - x_k)^2 f(\tilde{x}_k) f(\tilde{x}_i) + (t_i - t_k)(x_i - x_k)(f(\tilde{x}_k) + f(\tilde{x}_i)) \right)^2 \\
& \quad \left. - (t_i - t_k)^2 \left\{ [(x_i - x_k)f(\tilde{x}_k)]^2 + [(x_i - x_k)f(\tilde{x}_i)]^2 \right\} \right) \\
& + \psi_3(\|x_i - x_k\|) \left((t_i - t_k) \left[(t_i - t_k) + (x_i - x_k)(f(\tilde{x}_k) + f(\tilde{x}_i)) \right] [6 + 4f(\tilde{x}_k)f(\tilde{x}_i)] \right. \\
& \quad + f^2(\tilde{x}_i) \left\{ (x_i - x_k)f(\tilde{x}_k) + (t_i - t_k) \right\}^2 \\
& \quad + f^2(\tilde{x}_k) \left\{ (x_i - x_k)f(\tilde{x}_i) + (t_i - t_k) \right\}^2 \\
& \quad + [(x_i - x_k)^2 (f^2(\tilde{x}_k) + f^2(\tilde{x}_i)) \\
& \quad + 4(x_i - x_k)^2 f(\tilde{x}_k) f(\tilde{x}_i)] [1 + f(\tilde{x}_k) f(\tilde{x}_i)] \\
& \quad + \left\{ (t_i - t_k) + (x_i - x_k)f(\tilde{x}_k) \right\}^2 \left[(x_i - x_k)(f_x(\tilde{x}_i)f(\tilde{x}_i) + f_t(\tilde{x}_i)) \right] \\
& \quad \left. - \left\{ (t_i - t_k) + (x_i - x_k)f(\tilde{x}_i) \right\}^2 \left[(x_i - x_k)(f_x(\tilde{x}_k)f(\tilde{x}_k) + f_t(\tilde{x}_k)) \right] \right) \\
& + \psi_2(\|x_i - x_k\|) \left(3 + f^2(\tilde{x}_k) + f^2(\tilde{x}_i) \right. \\
& \quad + 4[f(\tilde{x}_k)f(\tilde{x}_i)] + 3[f(\tilde{x}_k)f(\tilde{x}_i)]^2 \\
& \quad - (1 + f^2(\tilde{x}_i)) \left[(x_i - x_k)(f_x(\tilde{x}_k)f(\tilde{x}_k) + f_t(\tilde{x}_k)) \right] \\
& \quad + (1 + f^2(\tilde{x}_k)) \left[(x_i - x_k)(f_x(\tilde{x}_i)f(\tilde{x}_i) + f_t(\tilde{x}_i)) \right] \\
& \quad - (x_i - x_k)^2 (f_x(\tilde{x}_k)f(\tilde{x}_k) + f_t(\tilde{x}_k))(f_x(\tilde{x}_i)f(\tilde{x}_i) + f_t(\tilde{x}_i)) \\
& \quad + 2 \left[(t_i - t_k) + (x_i - x_k)f(\tilde{x}_k) \right] \left[f(\tilde{x}_k)(f_x(\tilde{x}_i)f(\tilde{x}_i) + f_t(\tilde{x}_i)) \right] \\
& \quad \left. - 2 \left[(t_i - t_k) + (x_i - x_k)f(\tilde{x}_i) \right] \left[f(\tilde{x}_i)(f_x(\tilde{x}_k)f(\tilde{x}_k) + f_t(\tilde{x}_k)) \right] \right) \\
& - \psi_1(\|x_i - x_k\|) \left(f_x(\tilde{x}_k)f(\tilde{x}_k) + f_t(\tilde{x}_k) \right) \left(f_x(\tilde{x}_i)f(\tilde{x}_i) + f_t(\tilde{x}_i) \right).
\end{aligned}$$

Finally to approximate the area of exponential attraction we need to know the orbital derivative of the anstaz $w(x)$. To approximate this we introduce a new differential operator \mathcal{D} , it will be the orbital derivative so $\mathcal{D}w(\tilde{x}) = w'(\tilde{x}) = v_t(\tilde{x}) + v_x(\tilde{x}) \cdot f(\tilde{x})$, with $\tilde{x} \in \mathbb{I} \times \mathbb{R}$

$$w'(\tilde{x}) = \sum_{k=1}^N \alpha_k (\delta_{\tilde{x}} \circ \mathcal{D})^{\tilde{x}} (\delta_{\tilde{x}_k} \circ \mathcal{L})^{\tilde{y}} \Psi(\tilde{x} - \tilde{y}) + \sum_{k=1}^{N_1} \beta_k (\delta_{\tilde{x}} \circ \mathcal{D})^{\tilde{x}} (\delta_{\tilde{\eta}_k})^{\tilde{y}} \Psi(\tilde{x} - \tilde{y}),$$

which becomes,

$$\begin{aligned}
w'(\tilde{x}) = & \sum_{k=1}^N \alpha_k \left[\psi_3(\|\tilde{x} - \tilde{x}_k\|) \left((t - t_k)^3 + 2(t - t_k)^2(x - x_k)f(\tilde{x}_k) + (t - t_k)[(x - x_k)f(\tilde{x}_k)]^2 \right. \right. \\
& + (x - x_k)f(\tilde{x}) \left((t - t_k)^2 + 2(t - t_k)(x - x_k)f(\tilde{x}_k) \right) \\
& \left. \left. + (x - x_k)f(\tilde{x})[(x - x_k)f(\tilde{x}_k)]^2 \right) \right. \\
& + \psi_2(\|\tilde{x} - \tilde{x}_k\|) \left(3(t - t_k) + 2(x - x_k)f(\tilde{x}_k) + (t - t_k)f^2(\tilde{x}_k) \right. \\
& - (t - t_k)(x - x_k)[f_x(\tilde{x}_k)f(\tilde{x}_k) + f_t(\tilde{x}_k)] \\
& + (x - x_k)f(\tilde{x}) \left(1 + f^2(\tilde{x}_k) - (x - x_k)[f_x(\tilde{x}_k)f(\tilde{x}_k) + f_t(\tilde{x}_k)] \right) \\
& \left. \left. + 2f(\tilde{x}_k)f(\tilde{x}) \left((t - t_k) + (x - x_k)f(\tilde{x}_k) \right) \right) \right. \\
& \left. - \psi_1(\|\tilde{x} - \tilde{x}_k\|) \left(f(\tilde{x})[f_x(\tilde{x}_k)f(\tilde{x}_k) + f_t(\tilde{x}_k)] \right) \right] \\
& + \sum_{k=1}^{N_1} \beta_k \psi_1(\|\tilde{x} - \tilde{\eta}_k\|) \left((t - \tau_k) + (x - \eta_k)f(\tilde{x}) \right).
\end{aligned}$$

The linear system can be solved to determine the coefficient vector $(\alpha, \beta)^T$, then the coefficient vector can be used to construct $w'(\tilde{x})$. Then by using the approximation of $w'(\tilde{x})$ to determine level sets of $L_m(\tilde{x})$ one can approximate areas of exponential attraction.

3.3.2 Error Analysis

To show that the method is convergent we firstly make use of existing results on the standard error estimates for radial basis functions in approximating generalised linear differential equations found in [Giesl and Wendland, 2007] and presented in Chapter 2. We use Corollary 3.11 from [Giesl and Wendland, 2007] to give a modified version of the convergence proof, this version allows for the case where parts of the boundary, Γ , are not connected. The main measure for node distance we use is the fill distance,

$$h_G := \sup_{x \in G} \inf_{x_j \in X} \|x - x_j\|_2,$$

we also need the fill distance along the temporal boundary

$$h_\Gamma := \sup_{x \in \Gamma} \inf_{x_j \in X_\Gamma} \|x - x_j\|_2.$$

The following theorem from [Giesl and Wendland, 2007] with the boundary condition replaced with the one from Corollary 3.11 in [Giesl and Wendland, 2007] gives a general convergence result with respect to the fill distances, for the interior of the domain and on the temporal boundary.

Theorem 3.3.1. *Fix $\tau \in \mathbb{R}^+$ such that $k := \lfloor \tau \rfloor > 4$, $\sigma := \lceil \tau \rceil$ and $s := \tau - k$. Suppose Ψ generates a reproducing kernel Hilbert space that is norm equivalent to $W_2^\tau(\mathbb{I} \times \mathbb{R})$. Let $\Omega \in \mathbb{I} \times \mathbb{R}$ be a bounded domain having a piecewise $C^{k,s}$ -boundary. Finally, let w be the approximant to $W \in W_2^\tau(\Omega)$ satisfying (3.14).*

If the data have sufficiently small mesh-norms, then for all $1 \leq p \leq \infty$ the error estimates

$$\begin{aligned} \|LW - Lw\|_{L_p(\Omega)} &\leq Ch_G^{\tau-2-2(1/2-1/p)+} \|W\|_{W_2^\tau(G)}, \\ \|W - w\|_{L_p(\Gamma)} &\leq Ch_\Gamma^{\tau-1/2-(1/2-1/p)+} \|W\|_{W_2^\tau(\Omega)}, \end{aligned}$$

are satisfied.

The proof for this theorem is given in [Giesl and Wendland, 2007]. We show here that its application is appropriate for our situation. Indeed, the differential operator $LW = W''$ is of order 2 with coefficients $c_\alpha \in W_\infty^{k-1}(\Omega)$.

The above theorem can be applied to our approximation to give error bounds on the interior between w'' and W'' , and bounds between w and W on the boundary. The convergence result from the previous theorem can then be used to give an error bound for L_M and our approximation L_m , to do this we make use of a lemma from [Giesl, 2012].

Lemma 3.3.1. *Let $q, Q \in C^2(\mathbb{I})$ be functions with, $q(T_1) = a$, $q(T_2) = b$, $|q''(t)| \leq \epsilon$ for all $t \in (T_1, T_2)$, $Q(T_1) = A$, $Q(T_2) = B$, $Q''(t) = 0$ for all $t \in (T_1, T_2)$. Then we have for all $t \in [T_1, T_2]$*

$$|Q'(t) - q'(t)| \leq \frac{|A - a| + |B - b|}{T_2 - T_1} + \frac{3}{2}(T_2 - T_1)\epsilon, \quad (3.16)$$

$$|Q(t) - q(t)| \leq 2|A - a| + |B - b| + (T_2 - T_1)^2\epsilon, \quad (3.17)$$

$$Q'(t) = \frac{B - A}{T_2 - T_1}. \quad (3.18)$$

The proof comes from simple calculus, and comes from an earlier version of [Giesl, 2012].

Proof. By integration we obtain

$$q'(t) = \int_{T_1}^t q''(\tau) d\tau + c, \quad (3.19)$$

$$q(t) = \int_{T_1}^t \int_{T_1}^{\theta} q''(\tau) d\tau d\theta + c(t - T_1) + a, \quad (3.20)$$

where the second integration constant was chosen such that $q(T_1) = a$. As $q(T_2) = b$, from (3.20) we have

$$\begin{aligned} b &= q(T_2), \\ &= \int_{T_1}^{T_2} \int_{T_1}^{\theta} q''(\tau) d\tau d\theta + c(T_2 - T_1) + a, \text{ i.e.} \\ c &= \frac{b - a - \int_{T_1}^{T_2} \int_{T_1}^{\theta} q''(\tau) d\tau d\theta}{T_2 - T_1}. \end{aligned} \quad (3.21)$$

Note that $\int_{T_1}^{T_2} \int_{T_1}^{\theta} q''(\tau) d\tau d\theta = \int_{\Delta} q''(\tau) d\tau d\theta$, where Δ denotes the triangle $\Delta = \{(\theta, \tau) \in \mathbb{R}^2 \mid T_1 < \theta < T_2, T_1 < \tau < \theta\}$ of area $\frac{1}{2}(T_2 - T_1)^2$. Thus

$$\left| \int_{\Delta} q''(\tau) d\tau d\theta \right| \leq \frac{1}{2}(T_2 - T_1)^2 \epsilon.$$

We also integrate the function Q , taking the boundary conditions into account. We obtain

$$\begin{aligned} Q'(t) &= \frac{B - A}{T_2 - T_1}, \\ Q(t) &= \frac{B - A}{T_2 - T_1}(t - T_1) + A. \end{aligned}$$

This shows (3.18). Thus, we can calculate the following, using (3.19), (3.20) and (3.21),

$$\begin{aligned} |Q'(t) - q'(t)| &\leq (T_2 - T_1)\epsilon + \frac{|B - A - (b - a)|}{T_2 - T_1} + \frac{1}{2}(T_2 - T_1)\epsilon, \\ &\leq \frac{|A - a| + |B - b|}{T_2 - T_1} + \frac{3}{2}(T_2 - T_1)\epsilon. \\ |Q(t) - q(t)| &\leq \frac{1}{2}(T_2 - T_1)^2 \epsilon + \left| \frac{B - A}{T_2 - T_1} - c \right| (T_2 - T_1) + |A - a|, \\ &\leq (T_2 - T_1)^2 \epsilon + \frac{|B - b| + |A - a|}{T_2 - T_1} (T_2 - T_1) + |A - a|, \\ &= 2|A - a| + |B - b| + (T_2 - T_1)^2 \epsilon. \end{aligned}$$

Thus proving the lemma. \square

Using the Lemma 3.3.1 we now bound the error between our approximation and the true L_M .

Theorem 3.3.2. *Let $\mu(t)$ be an solution to (3.1) with rate of attraction γ_μ . Let L_m be the approximation to L_M constructed by the method laid out in this section, and let the mesh norms be such that $\|W''(t, x) - w''(t, x)\|_{L_\infty(G)} \leq \epsilon$ for all $(t, x) \in G$ and $\|W(t, x) - w(t, x)\|_{L_\infty(\Gamma)} \leq \delta$ for all $(t, x) \in \Gamma$ and $f \in C^k(\mathbb{I}, \mathbb{R})$ with $k \geq 3$ then*

$$|L_m(t, \mu(t)) - \gamma_\mu| \leq \frac{\delta}{T_2 - T_1} + \frac{3(T_2 - T_1)\epsilon}{4} \quad \text{for all } t \in \mathbb{I}.$$

Proof. Let $Q(t) := W(\mu(t)) + \int_{T_1}^t f_x(\tau, \mu(\tau))d\tau$, $q(t) := w(\mu(t)) + \int_{T_1}^t f_x(\tau, \mu(\tau))d\tau$, since $f \in C^k$ we have from Theorem 3.2.2 that $W \in C^{k-1}$. Then Q and q are functions that satisfy the properties of Lemma 3.3.1 with,

$$\begin{aligned} q(T_1) &= w(T_1, \mu(T_1)) + \int_{T_1}^{T_1} f_x(\tau, \mu(\tau))d\tau = a, \\ q(T_2) &= w(T_2, \mu(T_2)) + \int_{T_1}^{T_2} f_x(\tau, \mu(\tau))d\tau = b, \\ q''(t) &= w''(t, \mu(t)) + f'_x(t, \mu(t)). \end{aligned}$$

$$\begin{aligned} Q(T_1) &= W(T_1, \mu(T_1)) + \int_{T_1}^{T_1} f_x(\tau, \mu(\tau))d\tau = A, \\ Q(T_2) &= W(T_2, \mu(T_2)) + \int_{T_1}^{T_2} f_x(\tau, \mu(\tau))d\tau = B, \\ Q''(t) &= W''(t, \mu(t)) + f'_x(t, \mu(t)) = 0. \end{aligned}$$

As $W''(t, \mu(t)) = -f'_x(t, \mu(t))$ we have that

$$|q''(t)| = |w''(t, \mu(t)) - W''(t, \mu(t))| \leq \epsilon$$

Also as $|A - a| = |w(T_1, \mu(T_1)) - W(T_1, \mu(T_1))| \leq \delta$ as it is on the temporal boundary, the same holds for $|B - b| \leq \delta$. Then as $L_M(t, \mu(t)) = W'(t, \mu(t)) + f_x(t, \mu(t)) = \gamma_\mu$ we can

use this along with Lemma 3.3.1 to give,

$$\begin{aligned}
|L_m(t, \mu(t)) - \gamma_\mu| &= |L_M(t, \mu(t)) - L_m(t, \mu(t))|, \\
&= |W'(t, \mu(t)) - w'(t, \mu(t))|, \\
&= |Q'(t) - q'(t)|, \\
&\leq \frac{2\delta}{T_2 - T_1} + \frac{3(T_2 - T_1)}{2}\epsilon.
\end{aligned}$$

giving the desired result. \square

Remark

From Theorem 3.3.2 we have that $|L_m(t, \mu(t)) - \gamma_\mu| \leq \epsilon_*$. Therefore the approximation of the area of exponential attraction with maximal rate of attraction $-\nu$ will be given by the set $G_{\epsilon_*} := \{(t, x) | L_m(t, x) = -\nu - \epsilon_*\}$. This will not necessarily be an area of exponential attraction as there is no guarantee that G_{ϵ_*} will be invariant. However we can say that $G_{\epsilon_*} \subset G_{-\nu}$ where $G_{-\nu} := \{(t, x) | L_M(t, x) \leq -\nu\}$. This can be seen from the fact that $\{(t, x) | L_M(t, x) = -\nu\} \subset \{(t, x) | -\nu - \epsilon_* \leq L_m(t, x) \leq -\nu + \epsilon_*\}$.

3.4 Examples

3.4.1 Analytically Solvable System

We will consider the following example with $x \in \mathbb{R}$

$$\dot{x} = x(x - 1), \quad \mathbb{I} = [0, 1]. \quad (3.22)$$

We use the Radial Basis Function $\psi_{6,4}(2r)$ as seen in Table 2.2 to construct our approximation w of W , the solution of the boundary problem given in Corollary 3.3.2. On the boundary we use the following points $X_2 := \{(t, x) \mid t \in \{0, 1\}, x \in \{0, 0.05, 0.1, \dots, 0.8\}\}$. For the interior points we take $X_1 := \{(t, x) \mid t \in \{0.025, 0.075, 0.125, \dots, 0.975\}, x \in \{0, 0.05, 0.1, \dots, 0.8\}\}$. We show X_1 and X_2 as well as some numerically approximated solutions in Figure 3.1.

It is shown in [Giesl and Rasmussen, 2012] that the rate of exponential attraction for the

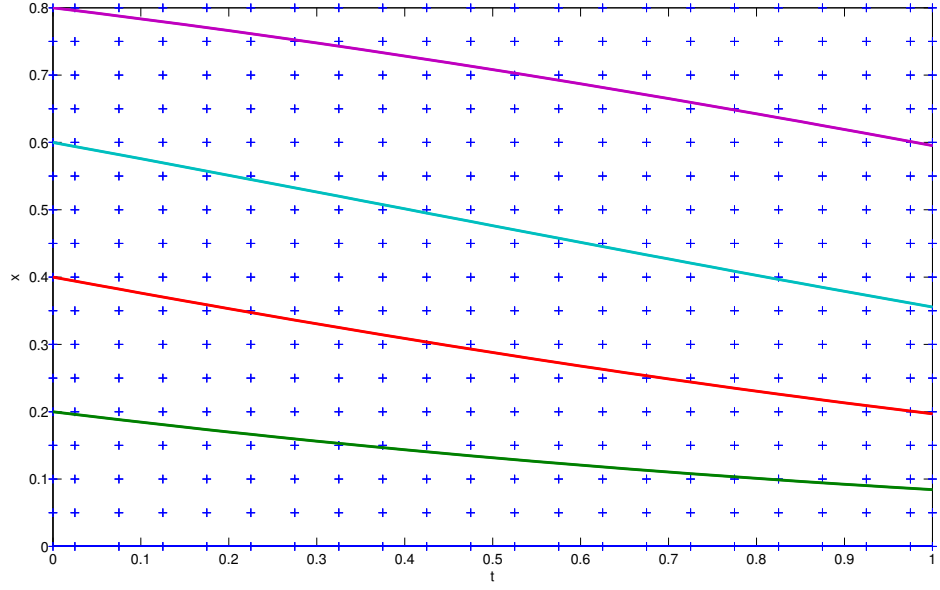


Figure 3.1: The sets of the interior points, X_1 and boundary points X_2 used in the construction of w are shown. Also shown are some solutions of the system given in (3.22).

solution $\varphi(t, 0, x_0)$ is given by

$$\gamma(x_0) := 1 - \frac{2}{T_2} \ln(x_0 - (x_0 - 1)e^{T_2}) \text{ for all } x_0 \in \left(-\infty, \frac{e^1}{e^1 - 1}\right).$$

where $T_2 = 1$. We use this form of γ and an inverse map of the solution to derive $L_M(t, x)$. This allows us to compare the approximation $L_m(t, x)$ directly with the function it is approximating. A plot of $L_m(t, x)$ is given in Figure 3.2.

In Figure 3.3 we produce a plot of the level sets of $L_m(t, x)$. In our approximation we have $L_m(0, 0.6229) \approx 0$. We define $\epsilon_0 := \frac{2\delta}{T_2 - T_1} + \frac{3(T_2 - T_1)}{2}$ where ϵ and δ are from Theorem 3.3.2. Then we determine that a subset of the area of exponential attraction is given by $\{(t, x) \mid t \in \mathbb{I} \text{ and } x \in \{x \mid L_m(t, x) \leq -\epsilon_0\}\}$. The true boundary of the area of exponential attraction is given by $\{(t, x) \mid t \in \mathbb{I} \text{ and } x \leq \varphi(t, 0, 0.\gamma_0)\}$ where $\gamma_0 = \frac{e - e^{1/2}}{e - 1} \approx 0.622459$. In Table 3.2 we show the error for different fill distances, the error was approximated on a grid with $h = 1/128$ and was defined as $\max_{(t,x) \in Y_{1/128}} \|W'(t, x) - w'(t, x)\|$, where $Y_{1/128}$ is the same as $X_1 \cup X_2$ except with a smaller fill distance. We took $\Psi_{6,4}$ as our Wendland function, $[0, 1]$ as our time domain and $[0, 0.8]$ as the space domain, the points were

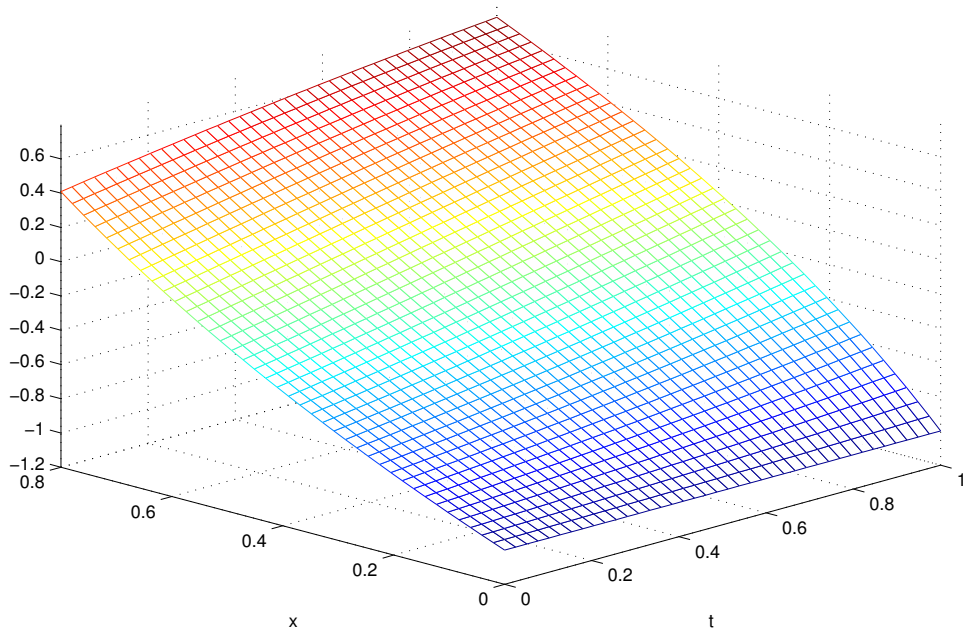


Figure 3.2: The approximation $L_m(t, x) = w'(t, x) + f_x(t, x)$.

uniformly placed in $[0, 1] \times [0, 0.8]$ and the support radius was $\frac{1}{2}$. There is some evidence to support a convergence rate of 2 however as Theorem 3.3.2 does not give a theoretical convergence rate in terms of the fill distance we cannot compare the numerical result.

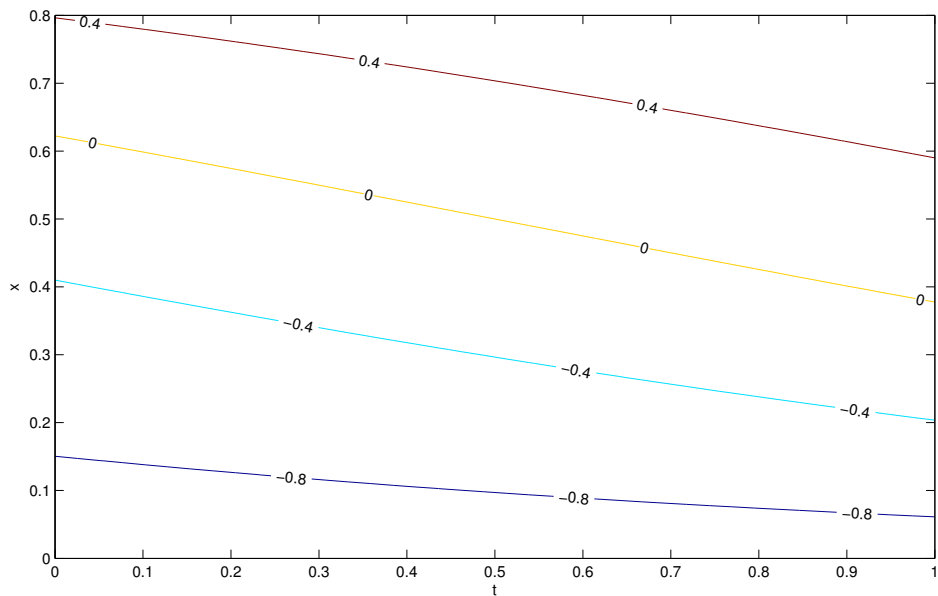


Figure 3.3: Level sets of $L_m(t, x) = L$, where $L = -0.8, -0.4, 0, 0.4$.

Table 3.1: Left: we give the $\max_{(t,x) \in Y_{1/128}} \|W'(t, x) - w'(t, x)\|$, the error was computed on a shifted mesh with $h = 1/128$. Right: We have numerically approximated convergence rates.

Table 3.2

h	$\ W'(t, x) - w'(t, x)\ _\infty$
1/8	0.2194
1/16	0.0893
1/32	0.0158
1/64	0.0080

Table 3.3

$\epsilon_{1/16}/\epsilon_{1/8}$	2.4579
$\epsilon_{1/32}/\epsilon_{1/16}$	5.6493
$\epsilon_{1/64}/\epsilon_{1/32}$	1.9789

3.4.2 Nonautonomous Example

For the second example we choose a nonautonomous Bernoulli type problem with $x \in \mathbb{R}$ on the time interval $[0, 2]$. It was used to approximate the domain of attraction in [Giesl, 2012]. It is particularly nice example as the distance between trajectories both increases and decreases along the time domain. The system is

$$\dot{x} = x \left(\frac{1}{4} - (t-1)^2 + x^3 \right), \quad \mathbb{I} = [0, 2]. \quad (3.23)$$

The solution is of the form,

$$\begin{aligned} &\varphi(t, 0, x_0) \\ &= x_0 \exp \left(-\frac{1}{3}t^3 + t^2 - \frac{3}{4} \right) \left(1 - 2x_0^2 \int_0^t \exp \left(-\frac{2}{3}\tau^3 + 2\tau^2 - \frac{3}{2}\tau \right) d\tau \right)^{-1/2}. \end{aligned}$$

Since $\int_0^t \exp \left(-\frac{2}{3}\tau^3 + 2\tau^2 - \frac{3}{2}\tau \right) d\tau$ cannot be analytically solved we cannot determine an analytic form of L_M . We have used the Radial Basis Function $\psi_{6,4}(0.5r)$ as seen in Table 2.2 to construct our approximation w of the solution to the boundary problem given in Corollary 3.3.2. For the boundary we use the following points $X_2 := \{(t, x) \mid t \in \{0, 2\}, x \in \{-0.32, -0.28, \dots, 0, 0.04, \dots, 0.32\}\}$. For the interior points we take $X_1 := \{(t, x) \mid t \in \{0.05, 0.15, 0.25, \dots, 1.95\}, x \in \{-0.32, -0.28, \dots, 0, 0.04, \dots, 0.32\}\}$. We show X_1 and X_2 as well as some numerically approximated solutions in Figure 3.4.

Figure 3.6 show some level sets of $L_m(\tilde{x})$. We define $\epsilon_0 := \frac{2\delta}{T_2 - T_1} + \frac{3(T_2 - T_1)}{2}$ where ϵ and δ are from Theorem 3.3.2. Then we determine that a subset of the area of exponential attraction is given by $\{(t, x) \mid t \in \mathbb{I} \text{ and } x \in \{x \mid L_m(t, x) \leq -\epsilon_0\}\}$. In this case we determine that the level set of $L(t, x) = 0$ is given by $x(0) \approx \pm 0.1789$. In Figure 3.7 we

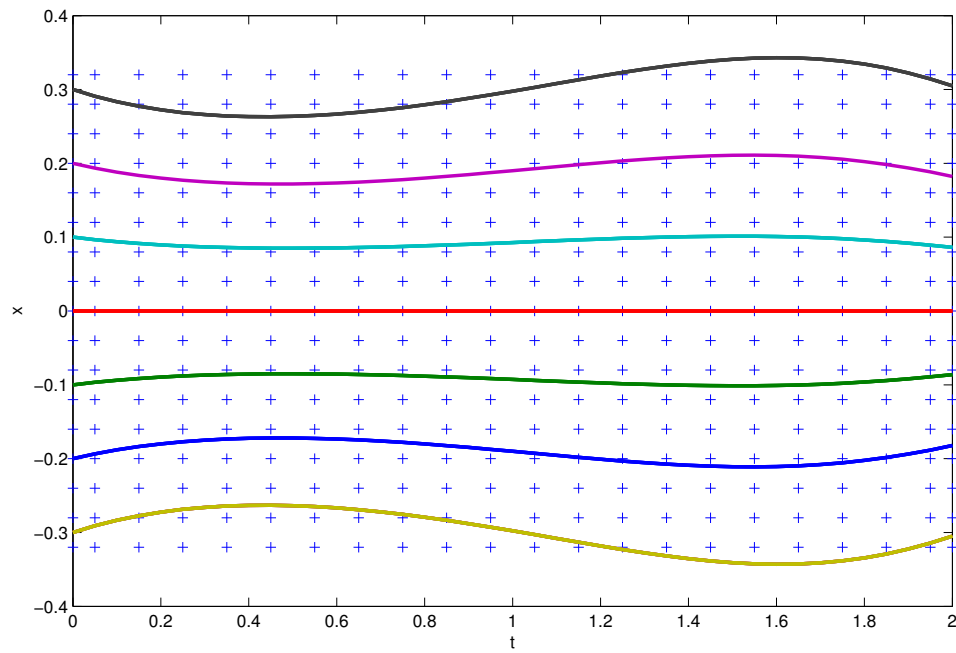


Figure 3.4: The sets of the interior points, X_1 and boundary points X_2 used in the construction of w are shown. Also shown are the some solutions of the system given in (3.23).

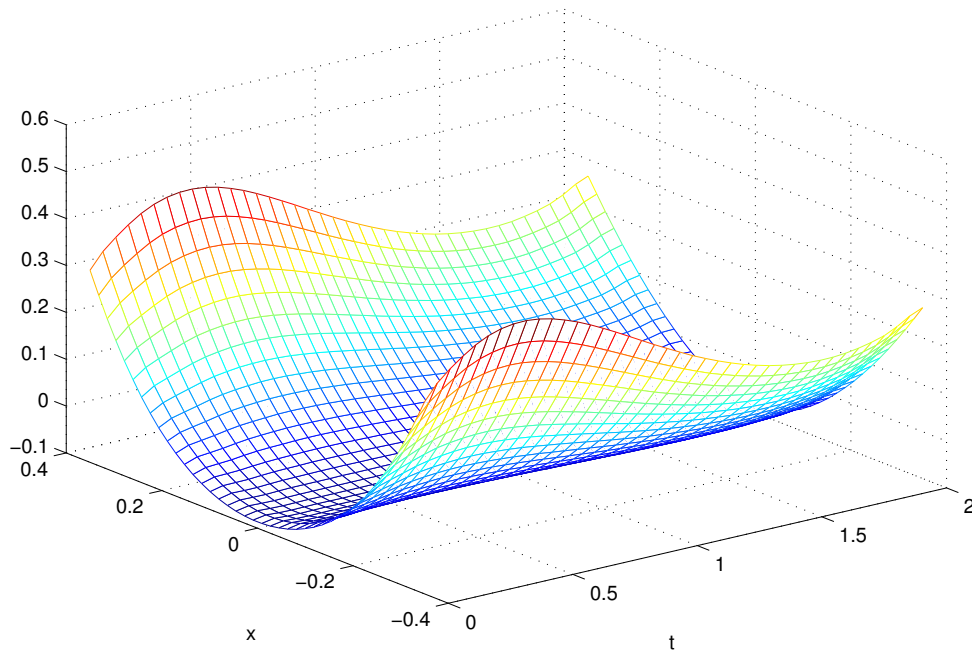


Figure 3.5: The approximation of $L_m(t, x)$

see that along this solution we closely mirror the zero-level set of L_m . Interestingly the area of exponential attraction is well contained by the domain of attraction as shown in

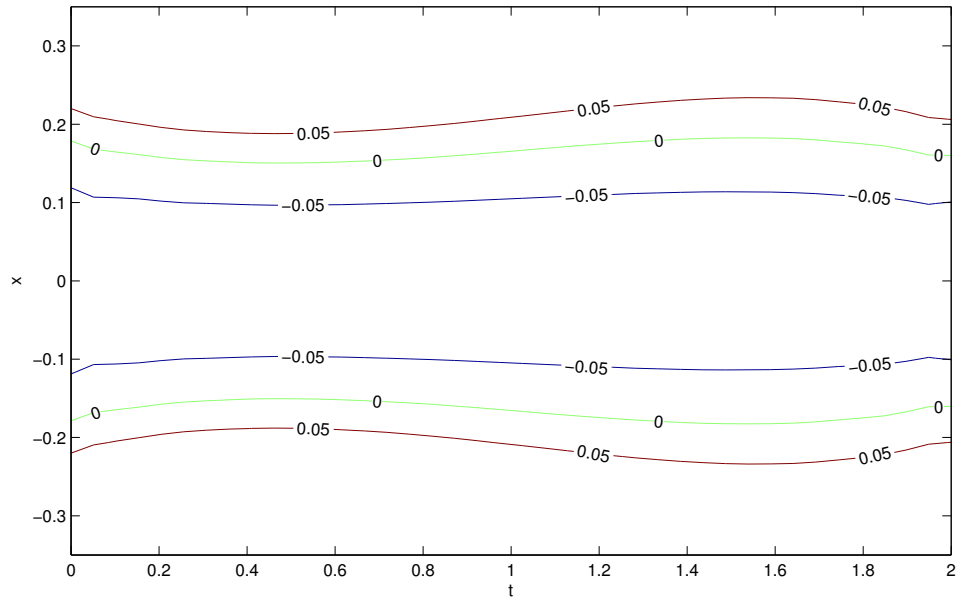


Figure 3.6: The level sets of $L_m(t, x)$. The 0 level has $x(0) \approx \pm 0.1789$. The true 0 level set of $L_M(t, x)$ will be in a small neighbourhood of the 0 level set of $L_m(t, x)$.

example 1 of [Giesl, 2012]. This is to be expected cf. Theorem 6.1 [Giesl and Rasmussen, 2012]. Furthermore when approximating the domain of attraction for a solution using the method given in [Giesl and Rasmussen, 2012] one cannot determine information about the domain of attraction in a small neighbourhood around the solution. However the method presented here does not have that problem so could be used to show that the neighbourhood around the solution lies in the domain of attraction for that solution.

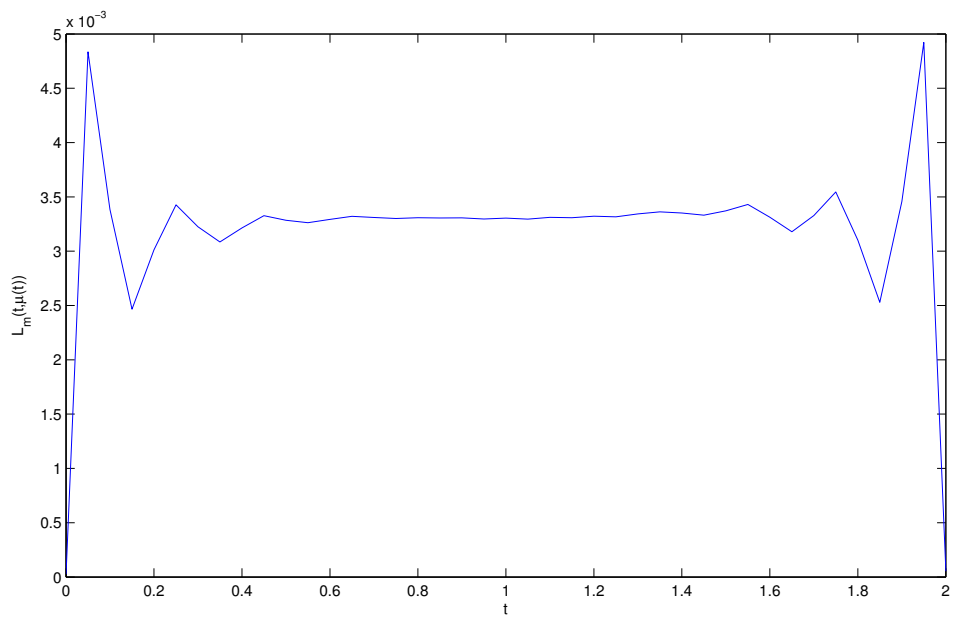


Figure 3.7: We show $L_m(t, \mu(t))$ where $\mu(t) = \varphi(t, 0, 0.1789)$. We numerically approximate $\mu(t)$ using Matlab's ODE45 function. $L_m(0, 0.1789) = 0$ to four significant figures.

Chapter 4

Approximating the basin of attraction of a periodic orbit in two dimensions

4.1 Introduction

This chapter shows a new method for approximating the basin of attraction of an exponentially stable periodic orbit in \mathbb{R}^2 . The method also gives the existence and uniqueness of that periodic orbit within a compact and positively invariant set. It extends the existing work [Giesl, 2007b], in which a method for determining the basin of attraction of an exponentially stable periodic orbit is given; that method also establishes both the uniqueness and existence of the orbit within a compact invariant set. The difference between the two methods is that in [Giesl, 2007b] the method requires attributes of the periodic orbit to be numerically approximated. Both methods have an advantage over the more traditional Lyapunov methods for determining basins of attraction, which require exact knowledge of where the periodic orbit is. The work in this chapter adds to the field a method for determining a subset of the basin of attraction of periodic orbit that does not require knowledge of any of the attributes of the orbit.

The work in this chapter is presented in two main parts. The first contains the theoretical foundation for the following sections. It shows that if a special contraction function is negative on a compact, positively invariant set that does not contain an equilibrium point,

it is both a necessary and sufficient condition for that set to be a subset of the basin of attraction of a unique exponentially stable periodic orbit. In the theorem that gives the necessity condition we derive an equation that will go on to be used to approximate the contraction function. This is not new work, it is based upon results in [Giesl, 2007b], [Giesl, 2004a] and [Giesl, 2003]; but it is an important foundation for why the methodology works. Where possible the author has paraphrased the work and simplified the mathematical methodologies, as the scope of this chapter is restricted to two dimensions.

Section 4.3 describes the new methodology that approximates the basin of attraction for the exponentially stable periodic orbit. This new methodology is a significant advancement in the field since, as far as the author is aware, it is the first method to approximate a subset of the basin of attraction that needs neither explicit knowledge of the location of the periodic orbit nor other attributes of the periodic orbit to be approximated. This allows it to be applied to systems where explicit knowledge of the orbit is not known. Section 4.3 also shows how the method is derived from the theoretical foundations covered in Section 4.2. In Section 4.4 we show that the error in the approximation of W' will decrease as denser meshes are used. Finally in Section 4.5 two examples are given: the first is a toy problem that can be solved analytically, and the second is a problem that can only be solved numerically and has been used previously in the existing literature.

4.2 Theoretical Foundations

In this chapter we continue to consider the autonomous ordinary differential equation

$$\dot{x} = f(x), \tag{4.1}$$

with $x \in \mathbb{R}^2$ and $f \in C^\sigma(\mathbb{R}^2, \mathbb{R}^2)$, with $\sigma \geq 1$ although further restrictions are specified later. We cite Theorem 1.1 from [Giesl, 2007b], which shows that given a positively invariant, compact, connected set K , if the distance between all adjacent trajectories of (4.1) contracts over time with respect to a Riemannian metric, then there exists a unique exponentially stable periodic orbit (Ω) in K . Furthermore K is a subset of the orbit's basin of attraction $(A(\Omega))$. This is a more specific version of Theorem 5 from [Giesl, 2004a] which uses a general Riemannian metric. Here the Riemannian metric is taken to

be to $M(p) = e^{2W(p)}$ where $W(p)$ is a scalar weight function, this is possible as we are in \mathbb{R}^2 . Below is the definition of an exponentially stable periodic orbit.

Definition 4.2.1. *Let S_t be the flow of a dynamical system given by an autonomous ordinary differential equation and let Ω be a periodic orbit. We will call Ω exponentially stable if, it is orbitally stable and there are $\delta, \mu > 0$ such that $\text{dist}(q, \Omega) \leq \delta$ implies $\text{dist}(S_t q, \Omega) e^{\mu t} \xrightarrow{t \rightarrow \infty} 0$.*

4.2.1 Sufficiency

In this subsection we present existing work that shows that a function L_W , that is made up of the traditional Euclidean contraction function (L) and the orbital derivative of a scalar weight function (W'), being negative on a compact, connected and positively invariant set is a sufficient condition for that set being a subset of the basin of attraction of a unique and exponentially stable periodic orbit. The following is Theorem 1.1 from [Giesl, 2007b].

Theorem 4.2.1. *Let $W \in C^1(\mathbb{R}^2, \mathbb{R})$ and let $\emptyset \neq K \subset \mathbb{R}^2$ be a compact, connected and positively invariant set, which contains no equilibrium. Moreover assume*

$$\max_{x \in K} L_W(x) := -\nu < 0, \text{ where}$$

$$L_W(x) := L(x) + W'(x),$$

$W'(x) = \langle \nabla W(x), f(x) \rangle$ denotes the orbital derivative and

$$L(x) = \frac{1}{\|f(x)\|^2} \begin{pmatrix} -f_2(x) & f_1(x) \end{pmatrix} Df(x) \begin{pmatrix} -f_2(x) \\ f_1(x) \end{pmatrix}.$$

Then there exists one and only one periodic orbit $\Omega \subset K$. The periodic orbit is exponentially stable, and the Floquet exponent different from 0 is less than or equal to $-\nu$. Moreover, $K \subset A(\Omega)$ holds, where $A(\Omega)$ denotes the basin of attraction of Ω .

The proof uses the general sufficiency result given in [Giesl, 2004a]. In that paper a more general version of Theorem 4.2.1 is given (Theorem 5) using a Riemannian metric. Corollary 14 in that paper goes on to show that if the metric is chosen to be a weight function of the form $e^{2W(p)}$ the theorem becomes Theorem 4.2.1. We define the function $f^\perp : \mathbb{R}^2 \rightarrow \mathbb{R}$ as $f^\perp(p) := \frac{1}{\|f(p)\|} (-f_2(p), f_1(p))^T$, (noting that it is well defined as we

exclude any equilibria in the set we are working with, hence $\|f(x)\| \neq 0$) and it is the normalised vector orthogonal to $f(p)$ i.e $\langle f^\perp(p), f(p) \rangle = 0$.

Remark

The proof of Theorem 4.2.1 is made up of a series of propositions and lemmas. We give an overview of them here.

Proposition 4.2.1 - shows that all points that are close to a point p that are orthogonal to the flow at p share an ω -limit set. It also defines a time diffeomorphism that synchronises the two trajectories so that they remain orthogonal for all future times. Finally it shows that the distance between the trajectories is decreasing at an exponential rate.

Lemma 4.2.1 - shows that all trajectories within a δ -neighbourhood of a point p will evolve within a bounded cone.

Lemma 4.2.2 - makes use of Lemma 4.2.1 to show that all trajectories close to a point p will be, or will have been orthogonal to the flow at p within a short amount of time.

Proposition 4.2.2 - makes use of Lemma 4.2.2 to show that any two points of K that are close to each other will share the same ω -limit set.

Proposition 4.2.3 - shows that since K is connected, all points in K share the same ω -limit set.

Proposition 4.2.4 - shows that the ω -limit set is a periodic orbit and that it is exponentially stable.

Finally - we close the proof of Theorem 4.2.1 by showing that $-\nu$ is an upper bound for the real part of the non-trivial Floquet exponent of the system.

The following proposition is the first step in proving Theorem 4.2.1. It establishes that for any point p inside K and any orthogonal point within a δ -neighbourhood of p there exists a time map that synchronises the velocities of the two trajectories. This means they remain orthogonal to one another as they travel through time with respect to the time map. It can then be shown that the distance between these two coordinated trajectories decreases exponentially with respect to time and they have the same ω -limit set. The following proposition comes from [Giesl, 2004a].

Proposition 4.2.1. *Let the assumptions of Theorem 4.2.1 be satisfied. Then for all $k \in (0, 1)$ there are constants $\delta > 0$ and $C \geq 1$ such that for all points $p \in K$ and for all $\eta \in \mathbb{R}^2$ with $\eta^T f(p) = 0$ and $\|\eta\| \leq \frac{\delta}{2}$ there exists a diffeomorphism $T_p^{p+\eta} : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ with $\frac{2}{3} \leq \dot{T}_p^{p+\eta}(\theta) \leq \frac{3}{4}$ that satisfies,*

$$(S_{T_p^{p+\eta}(\theta)}(p + \eta) - S_\theta(p))^T f(S_\theta(p)) = 0 \quad \text{for all } \theta \geq 0.$$

with $T_p^{p+\eta}$ being continuous with respect to η . Furthermore,

$$\|S_{T_p^{p+\eta}(\theta)}(p + \eta) - S_\theta(p)\| \leq C e^{-\nu(1-k)\theta} \|\eta\| \quad \text{for all } \theta \geq 0.$$

and,

$$\omega(p) = \omega(p + \eta).$$

To prove the proposition we start by introducing bounds on f and e^W . The time diffeomorphism (T) is then implicitly defined along with a distance metric that takes account of the weight function W . Bounds are then calculated for \dot{T} and it is shown that the distance between trajectories decreases exponentially with respect to time. Finally an argument is made to show that the two trajectories share the same ω -limit set.

Proof. Fix $p \in K$, as K is closed and bounded and does not contain any equilibrium points and f is continuous, the following holds,

$$0 < f_m \leq \|f(p)\| \leq f_M, \tag{4.2}$$

and since W is continuous there exist constants such that,

$$\begin{aligned} W_m &\leq e^{W(p)} \leq W_M, \\ \|W'(p)\| &\leq W_D \quad \text{both hold for all } p \in K. \end{aligned} \tag{4.3}$$

We define

$$\epsilon := \min \left(\frac{1}{3}, \frac{k}{3}, \frac{k\nu}{3W_D} \right). \tag{4.4}$$

Since Df is continuous it is uniformly continuous on K , hence there exists a $\delta_1 > 0$, such that for all $\xi \in \mathbb{R}^2$ with $\|\xi\| \leq \delta_1$ we have,

$$\|Df(p + \xi) - Df(p)\| \leq D_f := \frac{k\nu W_m^2}{4W_M^2}. \quad (4.5)$$

Finally, there is a positive constant f_D , such that,

$$\|Df(q)\| < f_D \text{ holds for all } q \in \overline{K_{\delta_1}}. \quad (4.6)$$

Let $F := \max(f_D, f_M)$, set $\delta := \min(\delta_1, \frac{\epsilon f_m^2}{4F^2}) \leq \frac{f_m^2}{2F^2}$ by definition of ϵ . We want T to be a map that synchronises the two trajectories' velocities, such that the distance remains orthogonal to $f(S_\theta p)$ for all future times. We define $T_p^{p+\eta}(\theta)$ implicitly as,

$$Q(T, \theta, \eta) := \langle S_T(p + \eta) - S_\theta(p), f(S_\theta(p)) \rangle = 0.$$

Furthermore as $Q(0, 0, \eta) = 0$, clearly $T_p^{p+\eta}(0) = 0$. Since we later show $\partial_T Q(0, 0, \eta) \neq 0$, the implicit function theorem gives that Q is defined locally near $\theta = 0$ and is continuous with respect to η . Next we define a distance mapping on the same region as that which Q is currently defined on. To ease with notation we write $T = T_p^{p+\eta}(\theta)$.

$$A : \begin{cases} \mathbb{R}_0^+ & \rightarrow \mathbb{R}_0^+ \\ \theta & \mapsto e^{W(S_\theta(p))} \|S_T(p + \eta) - S_\theta(p)\|. \end{cases}$$

Since $A(0) \neq 0$, as well as the uniqueness of solutions, this implies that $A(\theta) \neq 0$ for all $\theta \geq 0$. Therefore the vector $v(\theta) := \frac{S_T(p + \eta) - S_\theta(p)}{A(\theta)}$ is well defined and $\frac{1}{W_M} \leq \|v(\theta)\| \leq \frac{1}{W_m}$ holds for all $\theta \geq 0$.

From the implicit function theorem we have that, $\dot{T}_p^{p+\eta}(\theta) = -\frac{\partial_\theta Q(T, \theta, \eta)}{\partial_T Q(T, \theta, \eta)}$. Therefore to bound $\dot{T}_p^{p+\eta}$ the following are required,

$$\begin{aligned} \partial_\theta Q(T, \theta, \eta) &= -\|f(S_\theta(p))\|^2 + \langle S_T(p + \eta) - S_\theta(p), Df(S_\theta(p))f(S_\theta(p)) \rangle, \\ &= -\|f(S_\theta(p))\|^2 + A(\theta) \langle v(\theta), Df(S_\theta(p))f(S_\theta(p)) \rangle. \end{aligned}$$

Furthermore

$$\begin{aligned}
\partial_T Q(T, \theta, \eta) &= \langle f(S_T(p + \eta)), f(S_\theta(p)) \rangle, \\
&= \langle f(S_\theta(p) + A(\theta)v(\theta)), f(S_\theta(p)) \rangle, \\
&= \left(\|f(S_\theta(p))\|^2 + \left\langle A(\theta) \int_0^1 Df(S_\theta(p) + \lambda A(\theta)v(\theta)) d\lambda v(\theta), f(S_\theta(p)) \right\rangle \right),
\end{aligned}$$

with the mean value theorem giving the final line. Then we can see as stated earlier that $\partial_T Q(0, 0, \eta) = \langle f(p + \eta), f(p) \rangle \neq 0$ for sufficiently small η . Hence

$$\dot{T}_p^{p+\eta} = \frac{\|f(S_\theta(p))\|^2 - A(\theta) \langle v(\theta), Df(S_\theta(p)) \cdot f(S_\theta(p)) \rangle}{\|f(S_\theta(p))\|^2 + \left\langle A(\theta) \int_0^1 Df(S_\theta(p) + \lambda A(\theta)v(\theta)) d\lambda \cdot v(\theta), f(S_\theta(p)) \right\rangle}.$$

As $A(0) = e^{W(p)}\eta \leq W_M \frac{\delta}{2}$, and due to the continuity of A , there exists a T_0 such that $A(\theta) \leq W_M \delta$ for all $\theta \in [0, T_0]$. Moreover as $S_T(p + \eta) \in \overline{K_\delta}$ we can use (4.6) to give $\int_0^1 Df(S_\theta(p) + \lambda A(\theta)v(\theta)) d\lambda \leq f_D$. With this and (4.2), (4.3) and (4.5) we can bound \dot{T} by the following

$$\begin{aligned}
\dot{T}_p^{p+\eta}(\theta) &\leq \frac{\|f(S_\theta(p))\|^2 + \delta f_D f_M}{\|f(S_\theta(p))\|^2 - \delta f_D f_M}, \\
&= 1 + \frac{2\delta f_D f_M}{\|f(S_\theta(p))\|^2 - \delta f_D f_M}, \\
&\leq 1 + \frac{2\delta F^2}{f_m^2 - \delta F^2}, \\
&\leq 1 + \frac{2\delta F^2}{\frac{1}{2}f_m^2} \leq 1 + \epsilon \leq \frac{4}{3},
\end{aligned}$$

and

$$\begin{aligned}
\dot{T}_p^{p+\eta}(\theta) &\geq \frac{\|f(S_\theta(p))\|^2 - \delta f_D f_M}{\|f(S_\theta(p))\|^2 + \delta f_D f_M}, \\
&= 1 - \frac{2\delta F^2}{f_m^2 + \delta F^2}, \\
&\geq 1 - \frac{2\delta F^2}{f_m^2} \geq \frac{5}{7} \geq 1 - \epsilon \geq \frac{2}{3}.
\end{aligned}$$

Therefore $T_p^{p+\eta}(\theta)$ is a strictly increasing function. The inverse map $\theta(T)$ satisfies $\frac{3}{4} \leq \dot{\theta}(T) \leq \frac{3}{2}$. Hence if $A(\theta) \leq \delta W_M$ holds for all $\theta \geq 0$ then $T_p^{p+\eta}$ can be defined by a prolongation argument. To show that the two trajectories decrease exponentially the change of $A^2(\theta)$ with respect to θ is analysed; Cauchy-Schwartz and the same methods

are used as those used to bound $\partial_T Q$. We write $T_p^{p+\eta}(\theta)$ as $T(\theta)$ and note $A(\theta)v(\theta) = S_{T(\theta)}(p + \eta) - S_\theta(p)$ to aid with readability. Finally noting $f(S_\theta(p)) \perp v(\theta)$ then

$$\begin{aligned}
\frac{\partial A^2}{\partial \theta} &= 2W'(S_\theta(p))A^2(\theta) + 2e^{2W(S_\theta(p))}\langle f(S_{T(\theta)}(p + \eta))\dot{T}(\theta) - f(S_\theta(p)), S_{T(\theta)}(p + \eta) - S_\theta(p) \rangle, \\
&= 2W'(S_\theta(p))A^2(\theta) + 2e^{2W(S_\theta(p))}\dot{T}(\theta)\langle f(S_{T(\theta)}(p + \eta)), S_{T(\theta)}(p + \eta) - S_\theta(p) \rangle, \\
&= 2A^2(\theta)W'(S_\theta(p)) \\
&\quad + 2e^{2W(S_\theta(p))}\dot{T}(\theta)\left\langle f(S_\theta(p)) + \int_0^1 Df(S_\theta(p) + \lambda A(\theta)v(\theta))d\lambda A(\theta)v(\theta), A(\theta)v(\theta) \right\rangle, \\
&= 2A^2(\theta)\left(W'(S_\theta(p))(1 - \dot{T}(\theta))\right. \\
&\quad \left.+ 2\dot{T}e^{2W(S_\theta(p))}(\theta)\left\langle \int_0^1 [Df(S_\theta(p) + \lambda A(\theta)v(\theta)) - Df(S_\theta(p))]d\lambda A(\theta)v(\theta), A(\theta)v(\theta) \right\rangle\right. \\
&\quad \left.+ 2\dot{T}(\theta)A^2(\theta)(\langle Df(S_\theta(p))v(\theta), v(\theta) \rangle e^{2W(S_\theta(p))} + W'(S_\theta(p))).\right.
\end{aligned}$$

Since $S_\theta(p)$ is in K we can use (4.5) to bound the integral. Also, by its definition and since the assumptions of Theorem 4.2.1 are assumed to be true, $(\langle Df(S_\theta(p))v(\theta), v(\theta) \rangle e^{2W(S_\theta(p))} + W'(S_\theta(p)) = L_W(S_\theta(p)) \leq -\nu$. Therefore

$$\begin{aligned}
\frac{\partial A^2}{\partial \theta} &\leq 2A^2(\theta)\left(W_D\epsilon + (1 + \epsilon)\frac{W_M^2}{W_m^2}D_f - (1 - \epsilon)\nu\right), \\
&\leq 2A^2(\theta)\left(\frac{k\nu}{3} + \frac{k\nu}{3} + -nu\frac{k\nu}{3}\right), \\
&\leq -2(1 - k)\nu A^2(\theta).
\end{aligned}$$

Thus

$$A(\theta) \leq A(0)e^{-\nu(1-k)\theta} \leq W_M \frac{\delta}{2} e^{-\nu(1-k)\theta}. \quad (4.7)$$

This allows both $T(\theta)$ and $A(\theta)$ to be defined for all $\theta \geq 0$ by prolongation. Also

$$\begin{aligned}
W_m \|S_T(\theta)(p + \eta) - S_\theta p\| &\leq A(\theta), \\
&\leq A(0)e^{-\nu(1-k)\theta}, \\
&\leq W_M \|\eta\| e^{-\nu(1-k)\theta}.
\end{aligned}$$

Hence $C := \frac{W_M}{W_m} \geq 1$.

We now show that the ω -limit sets for p and $p + \eta$ are equal. Let $w \in \omega(p)$, then by

definition there exists a sequence θ_n with $\theta_n > \theta_{n-1}$, $\lim_{n \rightarrow \infty} \theta_n = \infty$ and $|w - S_{\theta_n}(p)| \xrightarrow{n \rightarrow \infty} 0$. Furthermore as $\dot{T}(\theta) \geq \frac{5}{7}$ for all $\theta \geq 0$ then $\lim_{n \rightarrow \infty} T(\theta_n) = \infty$ we have $\|S_{\theta_n}(p) - S_{T(\theta_n)}(p + \eta)\| \leq Ce^{-\nu(1-k)\theta_n} \rightarrow 0$. Hence $w \in \omega(p + \eta)$ and $\omega(p) \subset \omega(p + \eta)$. Conversely, assume $\hat{w} \in \omega(p + \eta)$, then there exists a strictly increasing sequence $\hat{\theta}_n \rightarrow \infty$ with a limit that satisfies $|\hat{w} - S_{\hat{\theta}_n}(p + \eta)| \rightarrow 0$. As the inverse map satisfies $\dot{T}^{(-1)}(\theta) \geq \frac{5}{7}$ for all $\theta \geq 0$ we have that $\hat{\theta}_n \xrightarrow{n \rightarrow \infty} \infty$. Therefore $\|S_{\hat{T}^{(-1)}(\hat{\theta}_n)}(p) - S_{\hat{\theta}_n}(p + \eta)\| \leq Ce^{-\nu(1-k)\hat{\theta}_n} \xrightarrow{n \rightarrow \infty} 0$ and $\hat{w} \in \omega(p)$ hence $\omega(p) = \omega(p + \eta)$. Thus we have proven Proposition 4.2.1. \square

To continue we need some results that describe the flow of trajectories q that lie within a δ -neighbourhood of p . We define a new coordinate system relative to the flow at p . Fix a $p \in K$, we define a hyperplane F^\perp which is a continuation of the line that contains $f^\perp(p)$. Let $\hat{f}(p) := \frac{f(p)}{\|f(p)\|}$, then $\langle f(p), \hat{f}(p) \rangle = \|f(p)\|$ hence $f_M \geq \langle f(p), \hat{f}(p) \rangle \geq f_m$. We define a new coordinate system x and y , which are defined as follows

$$\begin{aligned} y(q) &:= \langle q - p, \hat{f}(p) \rangle \in \mathbb{R}, \\ x(q) &:= q - p - y(q)\hat{f}(p) \in \mathbb{R}^2. \end{aligned}$$

Where y is the projection of $q - p$ onto the normalised flow at p and x is in the perpendicular direction to the flow at p , i.e. $\hat{f}^\perp(p)$. Rearranging yields an expression in terms of q , $q = p + y(q)\hat{f}(p) + x(q)$. Furthermore we can get an expression for $f(q)$ in terms of the new coordinate system

$$\begin{aligned} \lambda(q) &:= \langle f(q) - f(p), \hat{f}(p) \rangle, \\ w(q) &:= f(q) - f(p) - \lambda(q)\hat{f}(p), \end{aligned}$$

giving

$$f(q) = f(p) + \lambda(q)\hat{f}(p) + w(q). \quad (4.8)$$

Then by multiplying equation (4.8) by $\hat{f}(p)$ the following bound is derived

$$\langle f(q), \hat{f}(p) \rangle = \langle f(p), \hat{f}(p) \rangle + \langle \lambda(q)\hat{f}(p), \hat{f}(p) \rangle + \langle w(q), \hat{f}(p) \rangle,$$

using $w(q) \in f^\perp(p)$ and rearranging yields

$$\begin{aligned} |\langle \lambda(q) \hat{f}(p), \hat{f}(p) \rangle| &= |\langle f(q) - f(p), \hat{f}(p) \rangle|, \\ &\leq \|f(q) - f(p)\|. \end{aligned}$$

We now show that all trajectories starting in the δ -neighbourhood of p evolve within a cone. This result comes from the appendix of [Giesl, 2003].

Lemma 4.2.1. *Let $S_t(q) \in B_{\delta_4}(p)$ hold for all $t \in [-\tau_a, \tau_a]$ with a constant $\tau_a > 0$, where δ_4 is defined below. Then for all $t, \tau_1, \tau_2 \in [0, \tau_a]$ (or if $y(q) > 0$, $t, \tau_1, \tau_2 \in [-\tau_a, 0]$) with $\tau_1 < \tau_2$ the following hold*

$$\frac{1}{2}\|f(p)\| \leq \frac{d}{dt}y(S_t(q)) \leq \frac{3}{2}\|f(p)\|, \quad (4.9)$$

$$\frac{1}{2}\|f(p)\|(\tau_2 - \tau_1) \leq y(S_{\tau_2}(q)) - y(S_{\tau_1}(q)) \leq \frac{3}{2}\|f(p)\|(\tau_2 - \tau_1), \quad (4.10)$$

and

$$\|x(S_{\tau_2}q) - x(S_{\tau_1}q)\| \leq k_0(y(S_{\tau_2}q) - y(S_{\tau_1}q)), \quad (4.11)$$

where $k_0 := 4 \frac{W_M f_M}{W_m f_m}$.

The proof makes use of the new coordinate system to help create the new bounds.

Proof. As there is no equilibrium in K and f is continuous, let δ_4 be sufficiently small such that for all $q \in B_{\delta_4}(p)$ the following are true

$$\lambda(q) \leq \frac{1}{2}\|f(p)\|, \quad (4.12)$$

$$\|w(q)\| \leq \|f(p)\|. \quad (4.13)$$

Writing $S_t(q) = p + x(S_t(q)) + y(S_t(q))\hat{f}(p)$ then

$$\begin{aligned} f(S_t(q)) &= \frac{d}{dt}S_t(q), \\ &= \frac{d}{dt}y(S_t(q))\hat{f}(p) + \frac{d}{dt}x(S_t(q)). \end{aligned} \quad (4.14)$$

Since $\frac{d}{dt}x(S_t(q)) = f(S_t(q)) - \langle f(S_t(q), \hat{f}(p)) \rangle \hat{f}(p)$ and $\hat{f}(p)$ is normalised we see that

$$\begin{aligned} \left\langle \frac{d}{dt}x(S_t(q)), \hat{f}(p) \right\rangle &= \left\langle f(S_t(q)) - \langle f(S_t(q), \hat{f}(p)) \rangle \hat{f}(p), \hat{f}(p) \right\rangle, \\ &= \langle f(S_t(q), \hat{f}(p)) \rangle - \langle f(S_t(q), \hat{f}(p)) \rangle \|\hat{f}(p)\|^2, \\ &= 0, \end{aligned} \tag{4.15}$$

where the transition to the ultimate line was due to $\hat{f}(p)$ being normalised, hence its norm is one. Multiplying (4.14) by $\hat{f}(p)$ gives

$$\langle f(S_t(q), \hat{f}(p)) \rangle = \frac{d}{dt}y(S_t(q)),$$

resulting in

$$\begin{aligned} \|f(p)\| + \lambda(S_t(q)) &= \langle f(p), \hat{f}(p) \rangle + \langle f(S_t(q)) - f(p), \hat{f}(p) \rangle, \\ &= \frac{d}{dt}y(S_t(q)). \end{aligned}$$

Then (4.12) can be used to give (4.9) which can then be used along with $\int_{\tau_1}^{\tau_2} \frac{d}{dt}y(S_t(q))dt = y(S_{\tau_2}q) - y(S_{\tau_1}q)$ to give (4.10).

By rearranging (4.14), taking the inner product with $\frac{d}{dt}x(S_t(q))$, and using (4.15), the following is obtained

$$\begin{aligned} \left\| \frac{d}{dt}x(S_t(q)) \right\|^2 &= \left\langle \frac{d}{dt}x(S_t(q)), \frac{d}{dt}x(S_t(q)) \right\rangle, \\ &= \left\langle f(S_t(q)), \frac{d}{dt}x(S_t(q)) \right\rangle, \\ &= \left\langle f(p) + w(S_t(q)), \frac{d}{dt}x(S_t(q)) \right\rangle. \end{aligned}$$

Then applying Cauchy-Schwarz and dividing by $\left\| \frac{d}{dt}x(S_t(q)) \right\|$ gives

$$\left\| \frac{d}{dt}x(S_t(q)) \right\| \leq \|f(p)\| + \|w(S_t(q))\|. \tag{4.16}$$

Hence

$$\begin{aligned} \|x(S_{\tau_2}q) - x(S_{\tau_1}q)\| &\leq \int_{\tau_1}^{\tau_2} \left\| \frac{d}{dt} x(S_t q) \right\| dt, \\ &\leq 2(\tau_2 - \tau_1) \|f(p)\| \quad \text{by (4.13) and (4.16),} \end{aligned}$$

(4.11) follows. \square

The following is Lemma B.3 from [Giesl, 2003]. It shows that for all $p \in K$ and all points q within the δ -neighbourhood of p , there is a time when q 's trajectory will take it to a position where it is orthogonal to the flow at the point p . We also show that this will occur in a short amount of time. This lemma is used both in showing that all points within a δ -neighbourhood share an ω -limit set, and again later to show that the ω -limit set is an exponentially stable periodic orbit.

Lemma 4.2.2. *Let $\delta_2 := \frac{\delta_4}{\frac{4f_M}{3f_m} + 2}$. For all $q \in B_{\delta_2}(p)$ there exists a time τ_d with $|\tau_d| \leq t_M := \frac{2\delta_2}{3f_m}$. Moreover, there exists a continuous mapping π defined as follows*

$$\pi : \begin{cases} B_{\delta_2}(p) & \rightarrow B_{\delta_4}(p) \cap F^\perp, \\ q & \mapsto \pi(q), \end{cases}$$

such that $\pi(q) = S_{\tau_d}q$ and we define $F^\perp := \{p + v \mid v \perp f(p)\}$, where

$$\langle S_{\tau_d}(q) - p, f(p) \rangle = 0, \quad (4.17)$$

$$\|S_{\tau_d}(q) - p\| \leq (\kappa_0 + 1)\|q - p\|. \quad (4.18)$$

Proof. Firstly (4.17) is shown by finding a time $\tau_d(q)$ such that $y(S_{\tau_d(q)}) = 0$ and that τ_d is small enough such that for all τ between τ_d and 0, $S_\tau q$ does not leave the ball $B_{\delta_2}(p)$. We take the case when $y(q) > 0$ the other case works similarly and can be seen in [Giesl, 2003]. From (4.10) we have that $y(S_\tau q) \leq y(q) + \frac{3}{2}\tau\|f(p)\|$ for all $S_\tau q \in B_{\delta_4}(p)$, thus for $-\tau_* = \frac{2y(q)}{3\|f(p)\|}$ (note $|\tau_*| \leq \frac{2\delta_2}{3f_m} \leq t_M$), $y(S_{\tau_*}q) \leq 0$, then by the intermediate value theorem there exists a $\tau_d \in [\tau_*, 0]$ such that $y(S_{\tau_d}q) = 0$. As $y(S_\tau q)$ is monotonously increasing in τ , τ_d is unique and can be defined implicitly by $y(S_{\tau_d(q)}q) = 0$. As y and S_τ are continuous functions the continuity of τ^* and π follow from the implicit function

theorem.

As Lemma 4.2.1 was used we have to confirm that $S_t q \in B_{\delta_4}(p)$ for all $t \in [\tau_d, 0]$. This is shown by contradiction; assume there does exist a $t \in [\tau_d, 0]$ with $\|S_t q - p\| = \delta_4$ and let it be the first time that the trajectory leaves the ball, then

$$\begin{aligned} \delta_4 &= \|S_t q - p\|, \\ &= \left\| \int_0^t f(S_\tau q) d\tau + q - p \right\|, \\ &\leq \left\| \int_0^t f(S_\tau q) d\tau \right\| + \|q - p\|, \\ &\leq |\tau_d| f_M + \delta_2, \\ &\leq \delta_2 \left(\frac{2f_M}{3f_m} + 1 \right) = \frac{\delta_4}{2}. \end{aligned}$$

Hence we see a contradiction. Finally, from the definition of x we have that $\|S_{\tau_d} q - p\| = \|x(S_{\tau_d} q) + y(S_{\tau_d} q) \hat{f}(p)\|$. However $y(S_{\tau_d} q) = 0$ by definition, hence

$$\begin{aligned} \|S_{\tau_d} q - p\| &= \|x(S_{\tau_d} q)\|, \\ &\leq \|x(S_{\tau_d} q) - x(q)\| + \|x(q)\|, \\ &\leq \kappa_0 |y(q)| + \|x(q)\| \quad \text{by (4.11),} \\ &\leq (1 + \kappa_0) \|q - p\|. \end{aligned}$$

□

Now we can show that all points of K within a δ -neighbourhood of each other share the same ω -limit set. This is Proposition 8 from [Giesl, 2004a].

Proposition 4.2.2. *Let the assumptions of Theorem 4.2.1 be satisfied.*

Then there is a constant $\delta_ > 0$ such that $\omega(p) = \omega(q)$ holds for all $p \in K$ with $\|p - q\| \leq \delta_*$.*

Proof. Fix $p \in K$. Let δ_2 and δ be those from Lemma 4.2.2 and Proposition 4.2.1, respectively. Set $\delta_* := \min(\delta_2, \frac{\delta}{2(1 + \kappa_0)})$ and choose a q with $\|q - p\| \leq \delta_*$. Then from Lemma 4.2.2 we have that there exists a t_0 such that $\langle S_{t_0} q, p \rangle = 0$ and $\|S_{t_0} q - p\| \leq \delta/2$, hence Proposition 4.2.1 can be invoked to give $\omega(S_{t_0} q) = \omega(p)$ and thus $\omega(q) = \omega(p)$. □

Proposition 11 from [Giesl, 2004a] shows that all $p \in K$ share the same ω -limit set and

defines that set as Ω . The next step to prove theorem 4.2.1 is to show that any two points in K will have the same ω -limit set and it will equal the periodic orbit.

Proposition 4.2.3. *Let the assumptions of Theorem 4.2.1 be satisfied. Then $\emptyset \neq \omega(p) = \omega(q) =: \Omega \subset K$ for all $p, q \in K$.*

Proof. Let $p_0 \in K$. Since for all $\theta \geq 0$ we have $S_\theta p_0 \subset K$, which is compact set, $\emptyset \neq \omega(p_0) =: \Omega \subset K$.

Now consider an arbitrary point $p \in K$. By Proposition 4.2.2 we have $\omega(p) = \omega(q)$ for all q in a neighbourhood of p . Hence, $V_1 := \{p \in K | \omega(p) = \omega(p_0)\}$ and $V_2 := \{p \in K | \omega(p) \neq \omega(p_0)\}$ are open sets. Since $K = V_1 \cup V_2$, $p_0 \in V_1$ and K is connected V_2 must be empty and $V_1 = K$. \square

To prove that the ω -limit set is an exponentially stable periodic orbit if the conditions of Theorem 4.2.1 hold true we use Proposition B.1 from [Giesl, 2003].

Proposition 4.2.4. *Let $p \in \omega(p)$ and let p not be an equilibrium point. Assume there is a continuous map $g : \mathbb{R}_0^+ \rightarrow \mathbb{R}^2$ with $\|g(\theta)\|=1$ and $\langle g(\theta), f(S_\theta p) \rangle > 0$ for all $\theta \geq 0$. Moreover assume that there are constants $\delta, \nu > 0$ and $C \geq 1$ such that for all $\eta \in \mathbb{R}^2$ with $\eta \perp g(0)$ and $\|\eta\| \leq \delta$ there is a diffeomorphism $T_p^{p+\eta} : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ such that $T_p^{p+\eta}(\theta)$ depends continuously on η and satisfies $\frac{1}{2} \leq \dot{T}_p^{p+\eta} \leq \frac{3}{2}$ and*

$$\begin{aligned} \langle S_{T_p^{p+\eta}(\theta)}(p + \eta) - S_\theta p, g(\theta) \rangle &= 0, \\ \text{and } \|S_{T_p^{p+\eta}(\theta)}(p + \eta) - S_\theta p\| &\leq C e^{-\nu\theta} \|\eta\|, \end{aligned} \quad (4.19)$$

for all $\theta \geq 0$. Then p is a point of an exponentially asymptotically stable periodic orbit.

We do not prove Proposition 4.2.4 here, it can be found in the appendix of [Giesl, 2003]. We do give a quick heuristic explanation of the proof, many of the parts of it are similar (or the same) to other lemmas and propositions we have already used as a means to proving Theorem 4.2.1. Firstly a similar result to that of Lemma 4.2.1 is derived using the same new coordinate system. It shows that orbits within a δ -neighbourhood of p move within a bounded cone. The next step defines a hyperplane $p + g(0)^\perp$ and shows that all points close to p will cross that hyperplane in a short amount of time, this is similar in construction to Lemma 4.2.2. The next step is to create a Poincaré-like map that takes a point on the

hyperplane $p + g(0)^\perp$ and gives a time when the orbit intersects the hyperplane again. It is not a true Poincaré-map as the return time is not necessarily the first return-time. It is then shown that repeated application of the Poincaré-like map to a subset of the hyperplane results in a series of sets with decreasing diameter. This leads to the conclusion that there is a single point that lies in all these sets. By applying the Poincaré-like map to that single point \hat{p} there must exist a $T > 0$ such that $S_T \hat{p} = \hat{p}$ and \hat{p} is therefore a point on a periodic orbit. Then as both p and \hat{p} lie on the same hyperplane they will have the same ω -limit set, by Proposition 4.2.1. Hence $p \in \omega(p)$. By conditions of Proposition 4.2.1 all points close to p and on the hyperplane are attracted to $\omega(p)$ exponentially quickly, the final part of the proof is to show all points in the neighbourhood of $\omega(p)$ will cross the hyperplane in finite time.

In order to use Proposition 4.2.4 to prove that $\omega(p) = \Omega$ is an exponentially stable periodic orbit we take a point $p \in \Omega$, then from Proposition 4.2.3 we have that $\Omega = \omega(p)$, then take C from Proposition 4.2.1 and let $g(\theta) := f(S_\theta(p)) / \|f(S_\theta(p))\|$. Hence $\langle g(\theta), f(S_\theta p) \rangle = \|f(S_\theta p)\| > 0$. Hence Ω is an exponentially asymptotically stable periodic orbit and by Proposition 4.2.3 $\omega(q) = \Omega$ for all $q \in K$. Since Ω is asymptotically stable, $q \in A(\Omega)$ follows for all $q \in K$.

Finally to conclude the proof of Theorem 4.2.1 we show that $-\nu$ is an upper bound for the real part of the non-trivial Floquet exponent. We follow the methodology laid out in the proof of theorem 5 from [Giesl, 2004a], however we only need to show the case where the non-trivial Floquet exponent is real as we restrict ourselves to \mathbb{R}^2 .

Proof. Fix $p \in \Omega$ and define a hyperplane $H := \{p + v | \langle v, f(p) \rangle = 0\}$. Define a Poincaré map $\pi v := S_{T_p^{p+v}(T)}(p + v) - S_T(p)$, where T is the period of Ω , $T_p^{p+v}(T)$ is the time diffeomorphism as defined in Proposition 4.2.1 and v is sufficiently small. Next assume that the non-trivial Floquet exponent, $-\nu_0$, is real and $-\nu_0 > -\nu$. Then, for a sufficiently small $k > 0$ there exists $l > 0$ such that

$$e^{-\nu(1-k)T} = \sqrt{1-l} e^{-\nu_0 T}.$$

From [Hartman, 2002] we have that the non-trivial Floquet exponent is an eigenvalue of the linearised Poincaré map $D\pi$, therefore let u_o be a normalised eigenvector corresponding

to the eigenvalue $e^{-\nu_0}$ such that $D\pi u_0 = e^{-\nu_0} u_0$ and $\|u_0\| = 1$. Then by Taylor's theorem for $\epsilon > 0$, $\pi(\epsilon u_0) = e^{-\nu_0 T} \epsilon u_0 + k(\epsilon)$, is true where $\lim_{\epsilon \rightarrow 0} \|k(\epsilon)\|/\epsilon = 0$.

Given a vector $u := \epsilon u_0$ with $\epsilon \leq \delta/2$ then

$$\begin{aligned} \sqrt{1-l} e^{-\nu_0 T} e^{W(p)} \|u\| &= e^{-\nu(1-k)T} e^{W(p)} \|u\|, \\ &\geq A(T) \text{ (from (4.7))}, \\ &= e^{W(p)} \|\pi u\|, \\ &= e^{W(p)} \|e^{-\nu_0 T} \epsilon u_0 + k(\epsilon)\|. \end{aligned}$$

Then by dividing both sides by $e^{-\nu_0 T} \|\epsilon u_0\|$ we get

$$\sqrt{1-l} \geq \left\| \frac{u_0}{\|u_0\|} + k(\epsilon) \frac{e^{\nu_0 T}}{\|\epsilon u_0\|} \right\|,$$

then remembering that u_0 is normalised we get

$$\sqrt{1-l} \geq \left\| u_0 + k(\epsilon) \frac{e^{\nu_0 T}}{\epsilon} \right\|.$$

As $\epsilon \rightarrow 0$ the right hand side will tend to 1 which leads to a contradiction and hence the assumption that $-\nu_0 > -\nu$ is false. \square

Hence the proof of Theorem 4.2.1 is concluded and it has been shown that if we have a positively invariant, connected and compact set and can find a function such that $L_W(p) < 0$ for all points in the set then that set has a single exponentially stable periodic orbit and the set is a subset of the basin of attraction of the periodic orbit.

4.2.2 Existence of W

Next we want to show that the function W described in Theorem 4.2.1 exists and that L_W being less than zero is a necessary condition on the basin of attraction of an exponentially stable periodic orbit. Finally we seek to do this in such a way that we end up with a collocation equation for W which can be used to approximate it. The following theorem and its proof are from [Giesl, 2007b], we follow the layout introduced in [Giesl, 2007b]. Where possible, we have paraphrased and in some cases we have reworked the equation to take advantage of the scalar weight function.

Theorem 4.2.1. *Given the system $\dot{x} = f(x)$ with $f \in C^\sigma(\mathbb{R}^2)$ with $\sigma \geq 3$, let Ω be an exponentially stable periodic orbit with period T and Floquet exponents 0 and $-\nu < 0$. Define $F := \frac{1}{T} \int_0^T \|f(S_\tau p)\| d\tau$ with $p \in \Omega$, and we define $\mu := \frac{\nu}{F} > 0$. Then there exists a function $W \in C^{\sigma-2}(A(\Omega), \mathbb{R})$ such that*

$$W'(x) = -\mu\|f(x)\| - L(x), \quad \text{for all } x \in A(\Omega). \quad (4.20)$$

Overview of the proof:

I. New Coordinates and ν - we introduce a curvilinear coordinate system within the neighbourhood of Ω and show that $-\nu$ is equal to the integral of L on the orbit divided by the period of the orbit.

II. Projection - a map is introduced that projects points close to the orbit onto it.

III. Velocities - a map is introduced that matches velocities of trajectories with those on the orbit, this means that the trajectories have their velocities synchronised. Hence if a point q is mapped to $p \in \Omega$ by the projection map then the synchronised velocities of q and p means that at time t_1 , $S_{t_1}q$ is orthogonal to the point $S_{t_1}p$. This makes it much simpler to calculate the distance between trajectories as time moves forward as one always knows which point on the periodic orbit to measure against.

IV. Definition of s and g - s is a positive scalar value, it is the change in the time map from part III with respect to the speed on the periodic orbit, when the original system is multiplied by s it becomes a new system which has the matched velocities described in part III. A new differential system is established with g , which equals s times f .

V. Definition of W - W is firstly defined on the periodic orbit, it is then defined on the rest of U , where U is a small neighbourhood of the periodic orbit.

VI. Showing W is in $C^{\sigma-2}(U, \mathbb{R})$ - Lemma 4.2.3 gives the smoothness result on W .

VII. Returning to f - As s is a positive scalar it is shown that it can be removed from all sides of the equation and the equation for W no longer involves s and g .

VIII. Extending W to $A(\Omega)$ - A non-characteristic Cauchy equation is formed on all of $A(\Omega)$, W_{glob} is the global function W .

Proof. **I. New Coordinates and ν .**

Fix $p \in \Omega$. Introduce curvilinear coordinates (θ, n) with $x = S_\theta p + n f^\perp(S_\theta p)$ they are well defined within a neighbourhood of Ω , we define that neighbourhood as U .

We now establish a relation between L and ν . We have

$$\begin{aligned}
0 &= \frac{1}{2} \left(\ln \|f(S_T p)\|^2 - \ln \|f(p)\|^2 \right), \\
&= \frac{1}{2} \int_0^T \frac{d}{d\tau} \ln \|f(S_\tau p)\|^2 d\tau, \\
&= \frac{1}{2} \int_0^T \frac{1}{\|f(S_\tau p)\|^2} \frac{d}{d\tau} \|f(S_\tau p)\|^2 d\tau, \\
&= \int_0^T \frac{1}{\|f(S_\tau p)\|^2} \left\langle \frac{d}{d\tau} f(S_\tau p), f(S_\tau p) \right\rangle d\tau, \\
&= \int_0^T \frac{1}{\|f(S_\tau p)\|^2} \left[(f_{1x}(S_\tau p) f_1(S_\tau p) + f_{1y}(S_\tau p) f_2(S_\tau p)) f_1(S_\tau p) \right. \\
&\quad \left. + (f_{2x}(S_\tau p) f_1(S_\tau p) + f_{2y}(S_\tau p) f_2(S_\tau p)) f_2(S_\tau p) \right] d\tau, \\
&= \int_0^T \frac{1}{(f_1^2(S_\tau p) + f_2^2(S_\tau p))} \left[(f_{1x}(S_\tau p) f_1^2(S_\tau p) + f_{2y}(S_\tau p) f_2^2(S_\tau p) \right. \\
&\quad \left. + (f_{1y}(S_\tau p) + f_{2x}(S_\tau p)) f_1(S_\tau p) f_2(S_\tau p) \right] d\tau. \tag{4.21}
\end{aligned}$$

By definition we can write the trace as follows

$$\begin{aligned}
\int_0^T \text{tr} Df(S_\tau p) d\tau &= \int_0^T \frac{1}{(f_1^2(S_\tau p) + f_2^2(S_\tau p))} \left[f_{1x}(S_\tau p) (f_1^2(S_\tau p) + f_2^2(S_\tau p)) \right. \\
&\quad \left. + f_{2y}(S_\tau p) (f_1^2(S_\tau p) + f_2^2(S_\tau p)) \right] d\tau. \tag{4.22}
\end{aligned}$$

then by subtracting (4.21) from (4.22) we get the following

$$\begin{aligned}
\int_0^T \text{tr} Df(S_\tau p) d\tau &= \int_0^T \frac{1}{\|f(S_\tau p)\|^2} \left[f_2^2(S_\tau p) f_{1x}(S_\tau p) + f_1^2(S_\tau p) f_{2y}(S_\tau p) \right. \\
&\quad \left. - f_1(S_\tau p) f_2(S_\tau p) (f_{1y}(S_\tau p) + f_{2x}(S_\tau p)) \right] d\tau, \\
&= \int_0^T L(S_\tau p) d\tau. \tag{4.23}
\end{aligned}$$

We note since $z(t) = f(S_t p)$ is a solution of $\dot{z} = Df(S_t p)z$, 0 is a Floquet exponent of the system. Furthermore from [Hartman, 2002] we have that the sum of the Floquet exponents, $0 - \nu = \frac{1}{T} \int_0^T \text{tr} Df(S_\tau p) d\tau$. Combining this with (4.23) we have that $-\nu = \frac{1}{T} \int_0^T L(S_\tau p) d\tau$.

II. Projection.

Let $x \in U$. We then want to define a projection of x on the periodic orbit Ω such that

$P(x) = S_\theta p$ and $x - S_\theta P$ is orthogonal to the flow at $S_\theta p$. Hence we can define it implicitly as

$$\langle x - S_\theta p, f(S_\theta p) \rangle = 0. \quad (4.24)$$

Note that in the new coordinate system $P : U \rightarrow \Omega$ is defined by $P(\theta, n) = (\theta, 0)$, i.e. it will project any point in U back onto the periodic orbit. The implicit function theorem can be used to show that P is well defined near Ω and $P \in C^\sigma(U, \Omega)$. This can be seen by noting that

$$\frac{d}{d\theta} \langle x - S_\theta p, f(S_\theta p) \rangle = -\|f(S_\theta p)\|^2 + \langle x - S_\theta p, Df(S_\theta p)f(S_\theta p) \rangle. \quad (4.25)$$

When $x \in \Omega$ then the second term of (4.25) will be zero, and since $\|f(x)\| > 0$ for all $x \in \Omega$ there exists a neighbourhood of Ω where $\frac{d}{d\theta} \langle x - S_\theta p, f(S_\theta p) \rangle < 0$. Furthermore as Ω is stable we can ensure that the neighbourhood is sufficiently small that U is positively invariant.

III. Velocities.

Next we wish to create a way to synchronise the velocities of orbits in $U \setminus \Omega$ with those in Ω , such that given two points, one in $U \setminus \Omega$ and its projection onto Ω , then the evolution of flows is such that they remain orthogonal for all future times. This is achieved by defining a time map that synchronises the velocities of trajectories in a neighbourhood of the periodic orbit with the velocity of the solution on the periodic orbit, this is similar in construction to the diffeomorphism T that was defined in Proposition 4.2.1. This map will be different for each $x \in U$ so we fix $x \in U$ and let $q := P(x)$. $t = t(\theta)$ is then the function that ensures $P(S_t x) = S_\theta q$. The map $t(\theta) = t$ is implicitly defined by

$$Q(t, \theta) = \langle S_t x - S_\theta q, f(S_\theta q) \rangle = 0.$$

It is necessary to show that in a sufficiently small neighbourhood around Ω $Q_\theta(t, \theta) \neq 0$ holds, this allows the implicit function theorem to be used and gives that $t \in C^\sigma(\mathbb{R}, \mathbb{R})$. We want to establish that as time progresses - and hence the trajectory approaches Ω - that the synchronization will tend to 1. As $S_t x - S_\theta q = n f^\perp(S_\theta q)$ Taylor's theorem with

a Lagrange remainder gives the existence of a $\kappa \in (0, 1)$ such that $f(S_t x) = f(S_\theta q) + Df(S_\theta q + \kappa n f^\perp(S_\theta q)) n f^\perp(S_\theta q)$ and the implicit function theorem gives that

$$\begin{aligned}
\frac{dt}{d\theta} &= -\frac{Q_\theta(t, \theta)}{Q_t(t, \theta)}, \\
&= \frac{\|f(S_\theta q)\|^2 - \langle S_t x - S_\theta q, D(f(S_\theta q))f(S_\theta q) \rangle}{\langle f(S_t x), f(S_\theta q) \rangle}, \\
&= \frac{\|f(S_\theta q)\|^2 - n \langle f^\perp(S_\theta q), D(f(S_\theta q))f(S_\theta q) \rangle}{\|f(S_\theta q)\|^2 + \langle Df(S_\theta q + \kappa n f^\perp(S_\theta q)) n f^\perp(S_\theta q), f(S_\theta q) \rangle}, \\
&= 1 - n \frac{\langle f^\perp(S_\theta q), D(f(S_\theta q))f(S_\theta q) \rangle}{\|f(S_\theta q)\|^2 + \langle Df(S_\theta q + \kappa n f^\perp(S_\theta q)) n f^\perp(S_\theta q), f(S_\theta q) \rangle} \\
&\quad - n \frac{\langle Df(S_\theta q + \kappa n f^\perp(S_\theta q)) f^\perp(S_\theta q), f(S_\theta q) \rangle}{\|f(S_\theta q)\|^2 + \langle Df(S_\theta q + \kappa n f^\perp(S_\theta q)) n f^\perp(S_\theta q), f(S_\theta q) \rangle}, \\
&= 1 + O(n) \text{ as } n \rightarrow 0.
\end{aligned}$$

IV. Definition of s and g .

By multiplying the system (4.1) by $\frac{dt}{d\theta}$ on a sufficiently small neighbourhood of Ω we will get a system where the direction of the orbits is unchanged but the radial velocities are all 1. This allows us to state the distance between the trajectory and Ω without needing to calculate the nearest point on the periodic orbit, as that will be the evolved point that it was originally closest to. To achieve this we multiply the system by a scalar $s(x)$, defined as

$$s(x) := \frac{\|f(P(x))\|^2 - \langle x - P(x), Df(P(x))f(P(x)) \rangle}{\langle f(x), f(P(x)) \rangle},$$

as it is made of terms that all in $C^{\sigma-1}$ it is clear that $s \in C^{\sigma-1}(U, \mathbb{R})$ and for any $x \in \Omega$, $s(x) = 1$ hence it is possible to again reduce the size of the neighbourhood U so as to ensure $s(x) > 0$ for all $x \in U$. So we once again ensure that U is sufficiently small to meet this requirement and all previous ones as well. A new system is defined with $\dot{x} = s(x)f(x) =: g(x)$ where $g : U \rightarrow \mathbb{R}^2$. As s is positive scalar it will not alter the trajectories of the flows but just the velocities. In the construction of W , the function L

is used for the modified system, we call it L^g and it is given by the following

$$\begin{aligned}
 L^g(x) &= (g^\perp(x))^T Dg(x) g^\perp(x), \\
 &= (f^\perp(x))^T \left(s(x) Df(x) + f(x) \nabla s(x) \right) f^\perp(x), \\
 &= s(x) L^f(x),
 \end{aligned} \tag{4.26}$$

noting that $f^\perp(x) = g^\perp(x)$. As both s and L^f are in $C^{\sigma-1}$ then $L^g \in C^{\sigma-1}(U, \mathbb{R})$. We need to verify the claim that the radial velocity of all solutions is 1. By simple calculation it can be seen that $s(S_\theta q) = 1$ for $q \in \Omega$, hence

$$\begin{aligned}
 \frac{dt^g}{d\theta} &= \frac{\|g(S_\theta q)\|^2 - \langle S_t x - S_\theta q, Dg(S_\theta q) g(S_\theta q) \rangle}{\langle g(S_t x), g(S_\theta q) \rangle}, \\
 &= \frac{1}{s(S_t x)} \frac{\|f(S_\theta q)\|^2 - \langle S_t x - S_\theta q, Df(S_\theta q) f(S_\theta q) \rangle}{\langle f(S_t x), f(S_\theta q) \rangle}, \\
 &= 1.
 \end{aligned}$$

as by construction $P(S_t x) = S_\theta q$. This implies

$$P(S_t x) = S_t(P(x)) \quad \text{for all } x \in U \text{ and } t \geq 0. \tag{4.27}$$

Since $P(S_{t(\theta)} x) = S_\theta P(x)$ and $t^g(\theta) = \theta$. Hence for the new system $\dot{x} = g(x)$ defined on U the radial velocity is 1.

V. Definition of W .

We now construct $W(x)$ for $x \in U$, it is firstly done on the periodic orbit and then done for the rest of U . To be well defined on Ω it must be the case that $W(S_\theta p) = W(S_{\theta+T} p)$ for all $p \in \Omega$. Thus

$$W(S_\theta p) = \int_0^\theta \left[-\frac{\nu}{F} \|g(S_\tau p)\| - L(S_\tau p) \right] d\tau,$$

is well defined as $W(S_T p) = -\frac{\nu}{F} FT + T\nu = 0 = W(p)$, furthermore we have that

$$W'(S_\theta p) = -\frac{\nu}{F} \|g(S_\theta p)\| - L(S_\theta p).$$

So the condition holds on Ω . Next W is defined on U by the following

$$W(x) = \int_0^\infty \left[L(S_\tau x) - L(S_\tau P(x)) + \frac{\nu}{F} (\|g(S_\tau x)\| - \|g(S_\tau P(x))\|) \right] d\tau + W(P(x)). \quad (4.28)$$

We have,

$$\begin{aligned} W'(x) &= \frac{d}{dt} \lim_{T \rightarrow \infty} \int_0^T \left[L(S_{\tau+t} x) - L(S_{\tau+t} P(x)) + \frac{\nu}{F} (\|g(S_{\tau+t} x)\| - \|g(S_{\tau+t} P(x))\|) \right] d\tau \Big|_{t=0} + W'(P(x)), \\ &= \lim_{T \rightarrow \infty} \frac{d}{dt} \int_0^T \left[L(S_{\tau+t} x) - L(S_{\tau+t} P(x)) + \frac{\nu}{F} (\|g(S_{\tau+t} x)\| - \|g(S_{\tau+t} P(x))\|) \right] d\tau \Big|_{t=0} + W'(P(x)), \\ &= \lim_{T \rightarrow \infty} \frac{d}{dt} \int_t^{T+t} \left[L(S_\tau x) - L(S_\tau P(x)) + \frac{\nu}{F} (\|g(S_\tau x)\| - \|g(S_\tau P(x))\|) \right] d\tau \Big|_{t=0} + W'(P(x)), \\ &= \lim_{T \rightarrow \infty} \left[L(S_T x) - L(S_T P(x)) + \frac{\nu}{F} (\|g(S_T x)\| - \|g(S_T P(x))\|) \right] + W'(P(x)) \\ &\quad - (L(x) - L(P(x))) + \frac{\nu}{F} (\|g(x)\| - \|g(P(x))\|), \\ &= -L(x) - \frac{\nu}{F} \|g(x)\|. \end{aligned}$$

Where the following result was used along with the fact that $U \subset A(\Omega)$, then by using (4.27) we get

$$\begin{aligned} (W \circ P)'(x) &= \frac{d}{dt} W(P(S_t x)) \Big|_{t=0}, \\ &= \frac{d}{dt} W(S_t(P(x))) \Big|_{t=0}, \\ &= W'(P(x)). \end{aligned}$$

VI. Showing W is in $C^{\sigma-2}(U, \mathbb{R})$

To show that $W \in C^{\sigma-2}$ we use Lemma A.2. from [Giesl, 2007b].

Lemma 4.2.3. *The function W defined in (4.28) is in $C^{\sigma-2}(U, \mathbb{R})$. Moreover, $W'(x) = -\mu \|g(x)\| - L(x)$.*

The proof for this lemma can be found in the appendix of [Giesl, 2007b].

VII. Returning to f

Next we revert back to the original system, for the system $\dot{x} = f(x)$. We make use of the

fact that $s(x)$ is a scalar and $s(x) > 0$ for all $x \in U$

$$\begin{aligned} \langle \nabla W(x), g(x) \rangle &= -\frac{\nu}{F} \|g(x)\| - L^g(x), \\ \Leftrightarrow \langle \nabla W(x), s(x)f(x) \rangle &= -s(x)\frac{\nu}{F} \|f(x)\| - s(x)L(x), \\ \Leftrightarrow \langle \nabla W(x), f(x) \rangle &= -\frac{\nu}{F} \|f(x)\| - L(x). \end{aligned} \quad (4.29)$$

Where to move from the first to second line we used (4.26).

VIII. Extending W to $A(\Omega)$.

We now have an equation for W' but only in a small neighbourhood of Ω . We want to have an equation for W' on all of $A(\Omega)$. To do this we make use of Lemma A.1 from [Giesl, 2007b] that defines a noncharacteristic manifold Γ . We use this to define a noncharacteristic Cauchy problem.

Lemma 4.2.4. *Consider $\dot{x} = f(x)$ with $f \in C^\sigma(\mathbb{R}, \mathbb{R})$, $\sigma \geq 2$. Let Ω be an exponentially stable periodic orbit with Floquet exponents 0 and $-\nu < 0$.*

Then for all open neighbourhoods U of Ω there is a neighbourhood U_0 of Ω with $U_0 \subset U$ and a function $d \in C^{\sigma-1}(U_0 \setminus \Omega, \mathbb{R})$ such that its orbital derivative satisfies

$$d'(x) \leq -\frac{\nu}{2} \text{ for all } x \in U_0 \setminus \Omega.$$

Moreover, there is an $R \in \mathbb{R}$ such that $S = \{x \in U_0 \setminus \Omega \mid d(x) \leq R\} \cup \Omega$ is a positively invariant and compact set with $S \subset \overset{\circ}{U}_0$.

Denote $\Gamma = \{x \in U_0 \setminus \Omega \mid d(x) = R - 1\}$. There is a function $t \in C^{\sigma-1}(A(\Omega) \setminus \Omega, \mathbb{R})$ such that $S_t x \in \Gamma \Leftrightarrow t = t(x)$.

The global function W which we denote by W_{glob} is now the solution of the noncharacteristic Cauchy problem

$$\begin{aligned} \langle \nabla W_{glob}(x), f(x) \rangle &= -\mu \|f(x)\| - L(x) & \text{for } x \in A(\Omega) \setminus \Omega, \\ W_{glob}(x) &= W(x) & \text{for } x \in \Gamma. \end{aligned} \quad (4.30)$$

Where Γ is a noncharacteristic manifold as defined in Lemma 4.2.4. The characteristic equation of the linear partial differential equation $\langle \nabla W_{glob}(x), f(x) \rangle = -\mu \|f(x)\| - L(x)$

is the ordinary differential equation $\dot{x} = f(x)$. As there is no equilibrium of the system in $A(\Omega)$ there are no singular points. The characteristics meet every point in $A(\Omega) \setminus \Omega$. Thus there is a unique solution to W_{glob} to the Cauchy problem and, hence $W(x) = W_{glob}$ holds for all $x \in U \setminus \Omega$. Thus $W = W_{glob}$ such that $W \in C^{\sigma-2}(A(\Omega), \mathbb{R})$. This proves the theorem. \square

4.3 Approximating W

This is where we now diverge from existing work in the field and add our contribution. But before we do that we briefly describe how the approximation of the basin of attraction was constructed in [Giesl, 2007b]. This allows us to compare our method with the existing one. The author used equation (4.20) as the collocation-equation. He achieved this by numerically approximating μ , everything else on the right hand side is calculable. He then used $\mathcal{D}_1 w(x) := W'(x)$ as the differential operator for symmetric meshfree collocation. The ansatz with respect to a grid of distinct points $X_N = \{x_1, \dots, x_N\}$ becomes

$$\begin{aligned} w(x) &= \sum_{k=1}^N \beta_k (\delta_{x_k} \circ \mathcal{D}_1)^y \Psi(\|x - y\|), \\ &= \sum_{k=1}^N -\beta_k \psi_1(\|x - x_k\|) \langle x - x_k, f(x_k) \rangle, \end{aligned}$$

where we use the notation as described in Chapter 2. The j^{th} , k^{th} entry of the collocation matrix A will be

$$\begin{aligned} a_{j,k} &= (\delta_{x_j} \circ \mathcal{D}_1)^x (\delta_{x_k} \circ \mathcal{D}_1)^y \Psi(x - y), \\ &= -(\delta_{x_j} \circ \mathcal{D})^x \psi_1(\|x - x_k\|) \langle x - x_k, f(x_k) \rangle, \\ &= -\psi_2(\|x_j - x_k\|) \langle x_j - x_k, f(x_j) \rangle \langle x_j - x_k, f(x_k) \rangle \\ &\quad - \psi_1(\|x_j - x_k\|) \langle f(x_j), f(x_k) \rangle. \end{aligned}$$

Solving the collocation system to determine the coefficient matrix β_k allows for the recon-

struction of $W'(x)$, where the following equation is used

$$w'(x) = \sum_{k=1}^N \beta_k \left(-\psi_2(\|x - x_k\|) \langle x - x_k, f(x) \rangle \langle x - x_k, f(x_k) \rangle \right. \\ \left. - \psi_1(\|x - x_k\|) \langle f(x), f(x_k) \rangle \right).$$

We can then use the convergence results given in [Giesl, 2007b] and the methodology described above to determine a positively invariant set (see Section 4.6) to approximate a subset of the basin of attraction of a unique exponentially stable periodic orbit. However this is not the methodology that we pursue. By rearranging equation (4.20) and taking the orbital derivative of both sides it is possible to eliminate the term μ from the collocation equation. This means that no attributes of the periodic orbit will need to be numerically approximated to create the right hand side of the collocation equation. This advantage does come at the cost of requiring second order terms in the differential operator, the cost of which will be doubled by the symmetric collocation methodology. This leads to there being many terms in the collocation-equation. To show how this is done we introduce the following corollary.

Corollary 4.3.1. *Given the system $\dot{x} = f(x)$ with $f \in C^\sigma$ with $\sigma \geq 3$, let Ω be an asymptotically exponentially stable periodic orbit with period T and Floquet exponents 0 and $-\nu < 0$. Then there exists a function $W \in C^{\sigma-2}(A(\Omega), \mathbb{R})$ such that*

$$\mathcal{D}W(x) = L(x)\|f(x)\|' - L'(x)\|f(x)\|, \quad \text{for all } x \in A(\Omega). \quad (4.31)$$

Where \mathcal{D} is a differential operator with the following form

$$\mathcal{D}W(x) := W''(x)\|f(x)\| - W'(x)\|f(x)\|'. \quad (4.32)$$

Proof. From Theorem 4.2.1 we have

$$W'(x) = -\mu\|f(x)\| - L(x), \quad \text{for all } x \in A(\Omega). \quad (4.33)$$

Rearranging (4.33) and taking the orbital derivative gives

$$\left(\frac{W'(x)}{\|f(x)\|} \right)' = - \left(\frac{L(x)}{\|f(x)\|} \right)' - \mu'. \quad (4.34)$$

Note as μ is constant its orbital derivative is 0. Then from the quotient rule and multiplying through by $\|f(x)\|^2$ (which is greater than zero for all $x \in A(\Omega)$) this gives the following

$$W''(x)\|f(x)\| - W'(x)\|f(x)\|' = L(x)\|f(x)\|' - L'(x)\|f(x)\|. \quad (4.35)$$

□

We now give an explicit form for the differential operator defined in (4.32).

Lemma 4.3.1. *The differential operator $\mathcal{D}W(x) = W''(x)\|f(x)\| - W'(x)\|f(x)\|'$ for the autonomous ODE $\dot{x} = f(x)$ satisfies*

$$\begin{aligned} \mathcal{D}W(x) &= \|f(x)\|f^T(x)\text{Hess } W(x)f(x) + \|f(x)\|\nabla W^T(x)Df(x)f(x) \\ &\quad - \nabla W^T(x)f(x)\frac{f^T(x)Df(x)f(x)}{\|f(x)\|}. \end{aligned} \quad (4.36)$$

Proof. To prove this we take the second orbital derivative of W and then it is a case of bringing all the parts together

$$\begin{aligned} W'(x) &= \nabla W^T(x)f(x), \\ W''(x) &= \left\langle \nabla(\nabla W^T(x)f(x)), f(x) \right\rangle, \\ &= f^T(x)\text{Hess } W(x)f(x) + \nabla W^T(x)Df(x)f(x), \\ \|f(x)\|' &= \frac{f^T(x)Df(x)f(x)}{\|f(x)\|}. \end{aligned}$$

Where for W'' we have used that

$$\nabla(\nabla W \cdot f) \cdot f = \sum_{j=1}^2 \frac{\partial}{\partial x_j} \left(\sum_{i=1}^2 \frac{\partial W}{\partial x_i} f_i \right) f_j = \sum_{i,j=1}^2 \left(\frac{\partial^2 W}{\partial x_j \partial x_i} f_i f_j + \frac{\partial W}{\partial x_i} \frac{\partial f_i}{\partial x_j} f_j \right).$$

□

4.3.1 Building an Approximation to W

To construct an approximation of $W'(x)$ we will use symmetric meshfree collocation with radial basis functions as discussed in Chapter 2. We start by constructing the ansatz. Given a set of points $X_N = \{x_1, \dots, x_N\}$ with $x_i \in \mathbb{R}^2$, for $i = 1, \dots, N$ the ansatz is given by,

$$w(x) = \sum_{k=1}^N \alpha_k (\delta_{x_k} \circ \mathcal{D})^y \Psi(x - y).$$

By applying the differential operator to the ansatz at the grid points X_n we can construct a linear system and solve to determine the coefficients vector (α) . The the i^{th} row of the collocation system being given by

$$(\delta_{x_i} \circ \mathcal{D})w(x) = (\delta_{x_i} \circ \mathcal{D})^x W(x),$$

by using (4.35) this is equivalent to

$$w''(x_i) \|f(x_i)\| - w'(x_i) \|f(x_i)\|' = L(x_i) \|f(x_i)\|' - L'(x_i) \|f(x_i)\|,$$

for each $x_i \in X_N$. Then the following system is derived

$$A\alpha = R,$$

where $\alpha \in \mathbb{R}^{N \times 1}$ is a vector of unknown coefficients. $R \in \mathbb{R}^{N \times 1}$ where the i^{th} entry is given by $L'(x_i) \|f(x_i)\| - L(x_i) \|f(x_i)\|'$, the expanded form of this is given in (4.46). $A \in \mathbb{R}^{N \times N}$, with the i^{th} , k^{th} entry of A given by

$$A_{ik} = (\delta_{x_i} \circ \mathcal{D})^x (\delta_{x_k} \circ \mathcal{D})^y \Psi(x - y).$$

We give the expanded form of this in (4.45). We introduce a new differential operator \mathcal{D}_2 which is the second orbital derivative. Hence $\mathcal{D}_2 W := W''$. The ansatz with respect to

the set of points X_N

$$\begin{aligned} w(x) &= \sum_{k=1}^N \alpha_k (\delta_{y_k} \circ \mathcal{D})^y \Psi(x - y), \\ &= \sum_{k=1}^N \alpha_k \left[(\delta_{y_k} \circ \mathcal{D}_2)^y \Psi(x - y_k) \|f(y_k)\| - (\delta_{y_k} \circ \mathcal{D}_1)^y \Psi(x - y_k) \|f(y_k)\|' \right]. \end{aligned} \quad (4.37)$$

We will be making use of the following notation, which was introduced in Chapter 2

$$\psi_i(r) = \begin{cases} \frac{1}{r} \frac{\partial \psi_{i-1}(r)}{\partial r} & \text{for } r > 0, \\ 0 & \text{for } r = 0. \end{cases} \quad (4.38)$$

Furthermore, we have that

$$(\delta_y \circ \mathcal{D}_1)^y \Psi(x - y) = -\psi_1(\|x - y\|) \langle x - y, f(y) \rangle, \quad (4.39)$$

and

$$\begin{aligned} (\delta_y \circ \mathcal{D}_2)^y \Psi(x - y) &= \psi_2(\|x - y\|) \langle x - y, f(y) \rangle^2 \\ &\quad + \psi_1(\|x - y\|) \left(\|f(y)\|^2 - (x - y)^T Df(y) f(y) \right), \end{aligned} \quad (4.40)$$

where $Df(y)$ is the Jacobian of f . Finally

$$\|f(y)\|' = \frac{f^T(y) Df(y) f(y)}{\|f(y)\|}. \quad (4.41)$$

Bringing this together gives the ansatz with respect to the set of points X_N

$$\begin{aligned} w(x) &= \sum_{k=1}^N \alpha_k \left(\psi_2(\|x - y_k\|) \langle x - y_k, f(y_k) \rangle^2 \|f(y_k)\| \right. \\ &\quad \left. + \psi_1(\|x - y_k\|) \left(\|f(y_k)\|^3 + \left(\frac{\langle x - y_k, f(y_k) \rangle}{\|f(y_k)\|} f^T(y_k) - \|f(y_k)\| (x - y_k)^T \right) Df(y_k) f(y_k) \right) \right). \end{aligned} \quad (4.42)$$

To aid with layout we introduce the following notation

$$\psi_i := \psi_i(\|x - y_k\|), \quad (4.43)$$

$$M_k(x) := \left(\frac{\langle x - y_k, f(y_k) \rangle}{\|f(y_k)\|} f^T(y_k) - \|f(y_k)\| (x - y_k)^T \right) Df(y_k) f(y_k). \quad (4.44)$$

We bring this together to give an expanded for of w' . This will be used once the ansatz coefficients have been determined to approximation of L_M .

$$\begin{aligned} w'(x) = \sum_{k=1}^N \alpha_k \bigg[& \psi_3 \langle x - y_k, f(y_k) \rangle^2 \langle x - y_k, f(x) \rangle \|f(y_k)\| \\ & + 2\psi_2 \langle x - y_k, f(y_k) \rangle \langle f(y_k), f(x) \rangle \|f(y_k)\| \\ & + \psi_2 \langle x - y_k, f(x) \rangle \left(\|f(y_k)\|^3 + M_k(x) \right) \\ & + \psi_1 \left(\frac{\langle f(x), f(y_k) \rangle}{\|f(y_k)\|} f^T(y_k) - \|f(y_k)\| f^T(x) \right) Df(y_k) f(y_k) \bigg]. \end{aligned}$$

We now give the expanded form of the differential operator applied to the ansatz. It can be seen from the expanded form that $\mathcal{D}w(x)$ is symmetric.

$$\begin{aligned}
\mathcal{D}w(x) = & \sum_{k=1}^N \alpha_k \left\{ \psi_4 \langle x - y_k, f(x) \rangle^2 \langle x - y_k, f(y_k) \rangle^2 \|f(y_k)\| \|f(x)\| \right. \\
& + \psi_3 \left[4 \langle x - y_k, f(x) \rangle \langle x - y_k, f(y_k) \rangle \langle f(y_k), f(x) \rangle \|f(y_k)\| \|f(x)\| \right. \\
& \quad + \langle x - y_k, f(y_k) \rangle^2 \|f(y_k)\| \left(\|f(x)\|^3 \right. \\
& \quad \quad \left. - \left(\frac{\langle x - y_k, f(x) \rangle}{\|f(x)\|} f^T(x) - \|f(x)\| (x - y_k)^T \right) Df(x) f(x) \right) \\
& \quad + \langle x - y_k, f(x) \rangle^2 \|f(x)\| \left(\|f(y_k)\|^3 \right. \\
& \quad \quad \left. + \left(\frac{\langle x - y_k, f(y_k) \rangle}{\|f(y_k)\|} f^T(y_k) - \|f(y_k)\| (x - y)^T \right) Df(y_k) f(y_k) \right) \Big] \\
& + \psi_2 \left[2 \langle f(x), f(y_k) \rangle^2 \|f(y_k)\| \|f(x)\| + \|f(y_k)\|^3 \|f(x)\|^3 \right. \\
& \quad - \langle x - y_k, Df(x) f(x) \rangle \langle x - y_k, Df(y_k) f(y_k) \rangle \|f(y_k)\| \|f(x)\| \\
& \quad + 2 \langle x - y_k, f(y_k) \rangle \langle f(y_k), Df(x) f(x) \rangle \|f(y_k)\| \|f(x)\| \\
& \quad - 2 \langle x - y_k, f(x) \rangle \langle f(x), Df(y_k) f(y_k) \rangle \|f(y_k)\| \|f(x)\| \\
& \quad + \langle x - y_k, Df(x) f(x) \rangle \|f(y_k)\|^3 \|f(x)\| - \langle x - y_k, Df(y_k) f(y_k) \rangle \|f(y_k)\| \|f(x)\|^3 \\
& \quad + 2 \langle x - y_k, f(x) \rangle \langle f(y_k), f(x) \rangle \frac{f^T(y_k) Df(y_k) f(y_k)}{\|f(y_k)\|} \|f(x)\| \\
& \quad - 2 \langle x - y_k, f(y_k) \rangle \langle f(y_k), f(x) \rangle \frac{f^T(x) Df(x) f(x)}{\|f(x)\|} \|f(y_k)\| \\
& \quad + \langle x - y_k, f(y_k) \rangle \frac{f^T(y_k) Df(y_k) f(y_k)}{\|f(y_k)\|} \left(\langle x - y_k, Df(x) f(x) \rangle \|f(x)\| + \|f(x)\|^3 \right) \\
& \quad + \langle x - y_k, f(x) \rangle \frac{f^T(x) Df(x) f(x)}{\|f(x)\|} \left(\langle x - y_k, Df(y_k) f(y_k) \rangle \|f(y_k)\| - \|f(y_k)\|^3 \right) \\
& \quad - \langle x - y_k, f(x) \rangle \langle x - y_k, f(y_k) \rangle \frac{f^T(x) Df(x) f(x)}{\|f(x)\|} \frac{f^T(y_k) Df(y_k) f(y_k)}{\|f(y_k)\|} \Big] \\
& + \psi_1 \left(\langle Df(x) f(x), f(y_k) \rangle \frac{f^T(y_k) Df(y_k) f(y_k)}{\|f(y_k)\|} \|f(x)\| \right. \\
& \quad + \langle Df(y_k) f(y_k), f(x) \rangle \frac{f^T(x) Df(x) f(x)}{\|f(x)\|} \|f(y_k)\| \\
& \quad - \langle Df(y_k) f(y_k), Df(x) f(x) \rangle \|f(y_k)\| \|f(x)\| \\
& \quad \left. - \langle f(y_k), f(x) \rangle \frac{f^T(y_k) Df(y_k) f(y_k)}{\|f(y_k)\|} \frac{f^T(x) Df(x) f(x)}{\|f(x)\|} \right) \Big\}. \tag{4.45}
\end{aligned}$$

We give an expanded form of the right hand side of the collocation equation

$$\begin{aligned}
&= L(x)\|f(x)\|' - L'(x)\|f(x)\|, \\
&= 3 \frac{\left\langle f(x), Df(x)f(x) \right\rangle \left\langle f^\perp(x), Df(x)f^\perp(x) \right\rangle}{\|f(x)\|^3} \\
&\quad - \frac{1}{\|f(x)\|} \left[\left\langle Df^\perp(x)f(x), Df(x)f^\perp(x) \right\rangle + \left\langle (f^\perp(x))^T Df(x), Df^\perp(x)f(x) \right\rangle \right. \\
&\quad \left. + (f^\perp(x))^T \frac{\partial}{\partial x} Df(x)f^\perp(x)f_1(x) + (f^\perp(x))^T \frac{\partial}{\partial x} Df(x)f^\perp(x)f_2(x) \right]. \quad (4.46)
\end{aligned}$$

4.4 Error Analysis

In order to show that the method is convergent we use a the standard error result for radial basis functions and their application to dynamical systems as introduced in [Giesl and Wendland, 2007]. In Lemma 4.4.1 we show how the general convergence result can be utilised to show that the error between W' and the approximant w' is bounded by a constant and the mesh norm, i.e. it is convergent. Finally we show that our approximated version of L_m being negative on a positively invariant set is a sufficient condition for that set to be the subset of the basin of attraction of a unique exponentially stable periodic orbit.

Theorem 4.4.1. *Suppose Ψ is the reproducing kernel equivalent to $W_2^\tau(\mathbb{R}^n)$ with $k := \lfloor \tau \rfloor > m + \frac{n}{2}$. Let $K \subset \mathbb{R}^2$ be a bounded and compact domain having a $C^{k,s}$ -boundary and such that it does not include an equilibrium point of the system (4.1). Let \mathcal{D} be a linear differential operator of order m with coefficients c_α in $W_\infty^{k-m+1}(K)$. Finally let w be the generalised interpolant to $W \in W_2^\tau(K)$ from satisfying (4.35) as described in Section 4.3.*

If the data set has a sufficiently small mesh norm, then the error estimates

$$\|\mathcal{D}W - \mathcal{D}w\|_{L_\infty(K)} \leq Ch_{X_N, K}^{\tau-m-\frac{n}{2}} \|W\|_{W_2^\tau(k)}. \quad (4.47)$$

Proof. The proof of a generalised form of this theorem is given in [Giesl and Wendland, 2007]. Here we show that it is suitable to use for the collocation scheme described in Section 4.3. The differential operator given in (4.32) is of order 2, (i.e. $m = 2$) with coefficients c_α in $W^{k-1}(K)$ this can be seen from (4.36), K being bounded and that $\|f(x)\| > 0$ for $x \in K$. We also require $\tau \geq 3$ and therefore $f \in C^3$. Hence we can apply this result to

the collocation scheme described in this chapter. \square

We use the result from Theorem 4.4.1 to show that we can bound the error between our approximant and the true value for any $x \in K$, where K is a compact subset such that $K \subseteq A(\Omega)$. The methodology in this section tends to differ from the error analysis in other works in the field such as [Giesl, 2007b] and [Giesl, 2012] as well as Chapter 3. The difference is due to not being able to move between the differential operator and the orbital derivative via integration alone, due to our unusual differential operator. The process is also hampered by not having a boundary condition.

Lemma 4.4.1. *Let the assumptions of Theorem 4.4.1 be satisfied. Furthermore, let K be a positively invariant set and $\Omega \subset \overset{\circ}{K}$. For all $\epsilon > 0$ there exists a $h(\epsilon) > 0$ such that for all $w'(x)$ constructed as described in Section 4.3 with a mesh norm such that $h_{X_N, K} \leq h(\epsilon)$ we have*

$$\|W'(x) - w'(x)\| < \epsilon, \quad \text{for all } x \in K. \quad (4.48)$$

To prove this we start by showing that it holds true on the periodic orbit, this makes sense as in many respects the periodic orbit acts like a boundary term for this collocation problem. This is due to the functions having to conform to certain properties on it. We then show it holds true on the rest of K , to do this we fix a point $x \in K \setminus \Omega$, and denote t_1 as the first time when $\text{dist}(S_{t_1}x, \Omega) = \delta$. We then show that the error between W' and w' is bounded as the trajectories move towards $S_{t_1}x$.

Proof. **1. Preliminaries**

Since K is bounded, does not contain an equilibrium and f is continuous, there exist $f_m, f_M > 0$ such that $f_m \leq \|f(x)\| \leq f_M$ for all $x \in K$. T is the period of the orbit and $F := \frac{1}{T} \int_0^T \|f(S_t p)\| dt$ for $p \in \Omega$. Fix Ω_δ , a δ -neighbourhood of Ω , where δ is chosen such that the following holds true, note that $|\frac{d}{d\delta}\psi_1(\delta)|$ and $|\psi_2(\delta)|$ are bounded

$$|\psi_1(0)|f_D^2\delta^2 + 2f_M^2 \left| \max_{\hat{\delta} \in [0, \delta]} \frac{d}{d\delta}\psi_1(\hat{\delta}) \right| \left| \delta + 2f_M^2|\psi_2(\delta)|\delta^2 \right| \leq \left(\frac{\epsilon f_m}{3\|W\|_{\mathcal{F}} f_M} \right)^2. \quad (4.49)$$

Where $f_D > 0$ is a constant such that $\|Df(x)\| \leq f_D$ holds for all $x \in \overline{\text{conv}(K)}$, the closure of the convex hull of K . Define T_1 as the maximal time it takes all $x \in K$ to enter $\overline{\Omega_\delta}$.

Hence, $S_{T_1}x \in \overline{\Omega_\delta}$ for all $x \in K$.

1. Showing the result holds on Ω

Given an $\epsilon > 0$ we can use Theorem 4.4.1 and choose a sufficiently small $h(\epsilon)$ such that the right-hand side of (4.47) is smaller than $f_m^2 \epsilon_0$. Thus

$$\|f(x)\|^2 \left| \left(\frac{w'(x)}{\|f(x)\|} \right)' - \left(\frac{W'(x)}{\|f(x)\|} \right)' \right| \leq f_m^2 \epsilon_0. \quad (4.50)$$

Where $\epsilon_0 = \min \left(\frac{F f_m \epsilon}{3 f_M^2 T}, \frac{\epsilon}{3 f_M T_1} \right)$. Then we can obtain the following

$$\begin{aligned} \left| \left(\frac{w'(x)}{\|f(x)\|} \right)' - \left(\frac{W'(x)}{\|f(x)\|} \right)' \right| &\leq \frac{\epsilon_0 f_m^2}{\|f(x)\|^2}, \\ &\leq \epsilon_0. \end{aligned} \quad (4.51)$$

Using the definition of ϵ_0 gives

$$\left(\frac{w'(x)}{\|f(x)\|} \right)' \leq \left(\frac{W'(x)}{\|f(x)\|} \right)' + \frac{F \epsilon f_m}{3 f_M^3 T}.$$

Now we fix a point on the periodic orbit $p \in \Omega$ and show that $|W'(x) - w'(x)|$ can be bounded above for all $x \in \Omega$. We start by integrating along the solution from 0 to t with $t \in [0, T)$,

$$\frac{w'(S_t p)}{\|f(S_t p)\|} - \frac{w'(p)}{\|f(p)\|} \leq \frac{W'(S_t p)}{\|f(S_t p)\|} - \frac{W'(p)}{\|f(p)\|} + \frac{F \epsilon f_m}{3 f_M^3 T} t. \quad (4.52)$$

To aid in notation we define a variable $\hat{p}(w) := \frac{w'(p)}{\|f(p)\|} - \frac{W'(p)}{\|f(p)\|}$. Furthermore remembering from (4.20) that $W'(x) = -\mu\|f(x)\| - L(x)$ and that $t < T$ we have

$$\begin{aligned} w'(S_t p) &\leq W'(S_t p) + \frac{F \epsilon f_m \|f(S_t p)\|}{3 f_M^3} + \hat{p}(w) \|f(S_t p)\|, \\ &\leq -\mu \|f(S_t p)\| - L(S_t p) + \frac{F \epsilon f_m}{3 f_M^2} + \hat{p}(w) \|f(S_t p)\|. \end{aligned}$$

Now if we integrate along the entire periodic orbit and use $\frac{1}{T} \int_0^T L(S_t p) dt = -\nu$, we get

$$\begin{aligned} 0 &= \int_0^T w'(S_t p) dt \leq \int_0^T \left[-\mu \|f(S_t p)\| - L(S_t p) + \frac{F\epsilon f_m}{3f_M^2} + \hat{p}(w) \|f(S_t p)\| \right] dt, \\ &= -\mu TF + \nu T + \frac{FT\epsilon f_m}{3f_M^2} + \hat{p}(w)TF. \end{aligned}$$

As $\mu = \frac{\nu}{F}$ we get

$$-\frac{\epsilon f_m}{3f_M^2} \leq \hat{p}(w).$$

Equation (4.51) can also be rearranged to give

$$\left(\frac{w'(x)}{\|f(x)\|} \right)' \geq \left(\frac{W'(x)}{\|f(x)\|} \right)' - \frac{F\epsilon f_m}{3f_M^3 T}.$$

Following a similar process yields

$$\frac{\epsilon f_m}{3f_M^2} \geq \hat{p}(w).$$

This gives

$$w'(p) \leq W'(p) + \frac{\epsilon f_m}{3f_M},$$

and

$$w'(p) \geq W'(p) - \frac{\epsilon f_m}{3f_M}.$$

Hence for all $p \in \Omega$

$$|W'(p) - w'(p)| \leq \frac{\epsilon f_m}{3f_M}. \quad (4.53)$$

2. Showing the bound holds for the rest of K

Fix arbitrary $x \in K \setminus \Omega$ and denote t_1 with $t_1 \leq T_1$ as the first time that $\text{dist}(S_{t_1} x, \Omega) =$

$\|S_{t_1}x - p\| \leq \delta$ with suitable $p \in \Omega$. We wish to show that the following bound holds true

$$|w'(S_{t_1}x) - W'(S_{t_1}x) + W'(p) - w'(p)| \leq \frac{\epsilon f_m}{3f_M}. \quad (4.54)$$

Let \mathcal{D}_1 be the functional denoting the first order orbital derivative then define $\lambda := (\delta_{S_{t_1}x} \circ \mathcal{D}_1)$ and $\gamma := (\delta_p \circ \mathcal{D}_1)$ and note both $\lambda, \gamma \in \mathcal{F}^*$. Where \mathcal{F}^* is the dual of native space described in Section 2.8.

$$\begin{aligned} |w'(S_{t_1}x) - W'(S_{t_1}x) + W'(p) - w'(p)| &= |(\lambda - \gamma)(w - W)|, \\ &\leq \|\lambda - \gamma\|_{\mathcal{F}^*} \|w - W\|_{\mathcal{F}}, \\ &\leq \|\lambda - \gamma\|_{\mathcal{F}^*} \|W\|_{\mathcal{F}}. \end{aligned} \quad (4.55)$$

In (4.55) we have used the norm-minimal feature of the approximant in the native space described in Chapter 2. Then by following a similar methodology to the proof of Theorem 3.35 from [Giesl, 2007a] we can bound $\|\lambda - \gamma\|_{\mathcal{F}^*}$ by the following

$$\begin{aligned} \|\lambda - \gamma\|_{\mathcal{F}^*}^2 &= (\lambda - \gamma)^x (\lambda - \gamma)^y \Psi(x - y), \\ &= (\delta_{S_{t_1}x} \circ \mathcal{D}_1 - \delta_p \circ \mathcal{D}_1)^x (\delta_{S_{t_1}x} \circ \mathcal{D}_1 - \delta_p \circ \mathcal{D}_1)^y \Psi(x - y), \\ &\leq -\psi_1(0)(\|f(p)\|^2 + \|f(S_{t_1}x)\|^2) + 2\psi_1(\|p - S_{t_1}x\|)\langle f(p), f(S_{t_1}x) \rangle \\ &\quad - 2\psi_2(\|S_{t_1}x - p\|)\langle p - S_{t_1}x, f(p) \rangle \langle S_{t_1}x - p, f(S_{t_1}x) \rangle, \\ &\leq \psi_1(0)\|f(S_{t_1}x) - f(p)\|^2 \\ &\quad + 2[\psi_1(\|S_{t_1}x - p\|) - \psi_1(0)]\langle f(S_{t_1}x), f(p) \rangle \\ &\quad + 2|\psi_2(\|S_{t_1}x - p\|)|\|S_{t_1}x - p\|^2\|f(S_{t_1}x)\|\|f(p)\|. \end{aligned}$$

Denoting $r := \|S_{t_1}x - p\|$ and using Taylor's Theorem there are $\hat{r} \in [0, r]$ and $\eta = \theta S_{t_1}x + (1 - \theta)p$, where $\theta \in [0, 1]$ such that

$$\|\lambda - \gamma\|_{\mathcal{F}^*}^2 \leq |\psi_1(0)|f_D^2 r^2 + 2f_M^2 \left| \frac{d}{dr} \psi_1(\hat{r}) \right| r + 2f_M^2 |\psi_2(r)| r^2.$$

Furthermore due to the choice of δ made in (4.49) and $r \leq \delta$ we have

$$|\psi_1(0)|f_D^2 r^2 + 2f_M^2 \left| \frac{d}{dr} \psi_1(\hat{r}) \right| r + 2f_M^2 |\psi_2(r)| r^2 \leq \left(\frac{\epsilon f_m}{3\|W\|_{\mathcal{F}} f_M} \right)^2. \quad (4.56)$$

By using (4.56) in (4.55) we get that

$$|w'(S_{t_1}x) - W'(S_{t_1}x) + W'(p) - w'(p)| \leq \frac{\epsilon f_m}{3f_M}. \quad (4.57)$$

Now we show the bound holds true for the rest of K . From (4.51) the definition of ϵ_0 we have

$$\left| \left(\frac{w'(x)}{\|f(x)\|} \right)' - \left(\frac{W'(x)}{\|f(x)\|} \right)' \right| \leq \epsilon_0 \leq \frac{\epsilon}{3f_M T_1}. \quad (4.58)$$

Then

$$\int_0^{t_1} \left(\frac{w'(S_\tau x)}{\|f(S_\tau x)\|} \right)' d\tau \geq \int_0^{t_1} \left[\left(\frac{W'(S_\tau x)}{\|f(S_\tau x)\|} \right)' - \frac{\epsilon}{3f_M T_1} \right] d\tau.$$

This becomes,

$$\frac{w'(S_{t_1}x)}{\|f(S_{t_1}x)\|} - \frac{w'(x)}{\|f(x)\|} \geq \frac{W'(S_{t_1}x)}{\|f(S_{t_1}x)\|} - \frac{W'(x)}{\|f(x)\|} - \frac{\epsilon t_1}{3f_M T_1}. \quad (4.59)$$

Rearranging and multiplying through by $-\|f(x)\|$ gives

$$w'(x) \leq W'(x) + \frac{\epsilon T_1 \|f(x)\|}{3f_M T_1} + \frac{\|f(x)\|}{\|f(S_{t_1}x)\|} (w'(S_{t_1}x) - W'(S_{t_1}x)),$$

which then becomes $w'(x) - W'(x) \leq \frac{\epsilon}{3} + \frac{f_M}{f_m} |w'(S_{t_1}x) - W'(S_{t_1}x)|$. In a similar way we can show

$$|w'(x) - W'(x)| \leq \frac{\epsilon}{3} + \frac{f_M}{f_m} |w'(S_{t_1}x) - W'(S_{t_1}x)|. \quad (4.60)$$

We have

$$\begin{aligned} |w'(S_{t_1}x) - W'(S_{t_1}x)| &\leq |w'(S_{t_1}x) - W'(S_{t_1}x) + W'(p) + w'(p) - W'(p) - w'(p)|, \\ &\leq |w'(S_{t_1}x) - W'(S_{t_1}x) + W'(p) - w'(p)| + |w'(p) - W'(p)|. \end{aligned}$$

From (4.53) we have that $|w'(p) - W'(p)| \leq \frac{\epsilon f_m}{3f_M}$, and from (4.57) we have $|w'(S_{t_1}x) -$

$|W'(S_{t_1}x) + W'(p) - w'(p)| \leq \frac{\epsilon f_m}{3f_M}$. Using this we get

$$|w'(S_{t_1}x) - W'(S_{t_1}x)| \leq \frac{2\epsilon f_m}{3F_M}. \quad (4.61)$$

By putting (4.61) into (4.60) we have for arbitrarily chosen $x \in K$ that

$$|w'(x) - W'(x)| \leq \epsilon. \quad (4.62)$$

Hence the proof is complete. \square

Finally we show that the approximation of the generalised Borg-criterion L_w will be less than 0 on any compact set K such that $K \subset A(\Omega)$. The following proposition is from [Giesl, 2007b].

Proposition 4.4.1. *Let $k \in \mathbb{N}$, $k + \frac{7}{2} \leq \sigma$ and $f \in C^\sigma(\mathbb{R}^2, \mathbb{R}^2)$. Let Ω be a periodic orbit with Floquet exponents 0 and $-\nu < 0$ and let $K \subset A(\Omega)$ be positively invariant and a compact set with $\Omega \subset \overset{\circ}{K}$. Prolongate the function $W \in C^{\sigma-2}(A(\Omega), \mathbb{R})$ of Theorem 4.2.1 to a function $W \in C_0^{\sigma-2}(\mathbb{R}^2, \mathbb{R})$ which coincides with the former W on K . Define $\epsilon^* := \min \left(1, \frac{\mu}{2} \min_{x \in K} \|f(x)\| \right)$, using the notations of Lemma 4.4.1. Let $h(\epsilon^*)$ be a mesh norm that is sufficiently small to satisfy (4.48) from Lemma 4.4.1 with ϵ^* . Let $X_N \subset K$ be a grid with a fill distance $h \leq h(\epsilon^*)$ in K . Then the reconstruction w of W with respect to the grid X_N satisfies*

$$L_w(x) < 0 \text{ for all } x \in K.$$

Proof. We apply Lemma 4.4.1 to W . Note that $W \in C_0^{k+3/2}(\mathbb{R}^2, \mathbb{R})$. Hence by (4.48) we have $w'(x) \leq W'(x) + \epsilon^* \leq -\mu\|f(x)\| - L(x) + \epsilon^* \leq -\mu\|f(x)\| - L(x) + \frac{\mu}{2} \min_{x \in K} \|f(x)\|$. Hence for all $x \in K$, we have

$$\begin{aligned} L_w(x) &= w'(x) + L(x), \\ &\leq -\mu\|f(x)\| + \frac{\mu}{2} \min_{x \in K} \|f(x)\|, \\ &\leq -\frac{\mu}{2} \min_{x \in K} \|f(x)\| \\ &< 0. \end{aligned}$$

□

4.5 Examples

We show the method working on two systems, the first is a very simple system for which we can derive an analytic version of $L_M(x)$. This allows us to calculate the error in the approximation. The second is the Van-der-Pol system, we cannot derive an analytic solution for this, but we have constructed the problem in such a way that we can make direct comparisons to the example given in [Giesl, 2007b].

4.5.1 Analytic Example

In this example we work on a simple dynamical system to show that the method works.

The system we work with is

$$\begin{cases} \dot{x} = x(1 - (x^2 + y^2)) - y, \\ \dot{y} = y(1 - (x^2 + y^2)) + x. \end{cases} \quad (4.63)$$

For the rest of this section we introduce the notation $\mathbf{x} = (x, y)^T$. The system has an equilibrium at the origin and an exponentially stable periodic orbit $\Omega := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, $A(\Omega) = \mathbb{R}^2 \setminus \{0\}$. For the analytic reconstruction of $W'(\mathbf{x})$ we need to know both $\mu = 2$ and $L(\mathbf{x})$. $L(\mathbf{x})$ is given by

$$L(\mathbf{x}) = \frac{2 - 6\|\mathbf{x}\|^5 + 3\|\mathbf{x}\|^8 - \|\mathbf{x}\|^{12}}{1 + (1 - \|\mathbf{x}\|^4)^2}.$$

Therefore we know the true value of W' and it is given by

$$W'(\mathbf{x}) = -2\|f(\mathbf{x})\| - L(\mathbf{x}).$$

We define $K := \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \geq 0.5 \text{ and } \|\mathbf{x}\| \leq 1.5\}$. It is clear that $\Omega \subset K$ and K is positively invariant. The collocation was performed on a set grid points X_N that were placed in a hexagonal grid as was shown is optimal in [Iske, 1999]. The fill distance is $h = 0.1 \frac{\sqrt{5}}{2}$. The grid points can be seen in Figure 4.1 along with the Ω and four sample trajectories. The reconstruction was performed with the radial basis function $\psi_{6,4}\left(\frac{1}{5.5}r\right)$ which is defined in Table 2.2. In Figure 4.2 we show some level sets of $L_m(\mathbf{x})$, from this

one can see that $L(\mathbf{x}) < 0$ for all $\mathbf{x} \in K$ and hence we have numerically confirmed that $K \subset A(\Omega)$. In Figure 4.3, $L_m(\mathbf{x}) - L_M(\mathbf{x})$ is shown some data points that are in K but not in X_N .

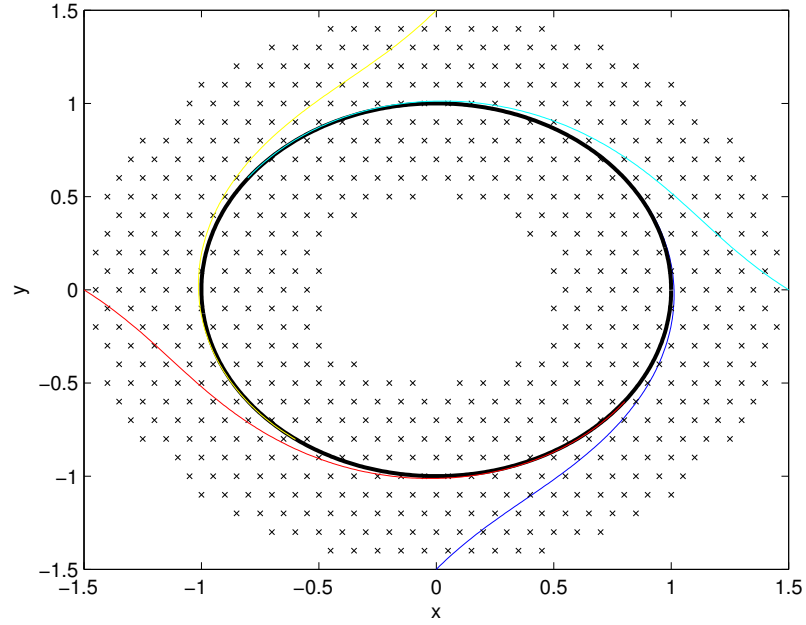


Figure 4.1: The grid for the approximation, the fill distance is $h = 0.1 \frac{\sqrt{5}}{2}$. Also shown are Ω (black) and the solutions with initial positions $(-1.5, 0)^T$ (red), $(1.5, 0)^T$ (cyan), $(0, -1.5)^T$ (blue) and $(0, 1.5)^T$ (yellow).

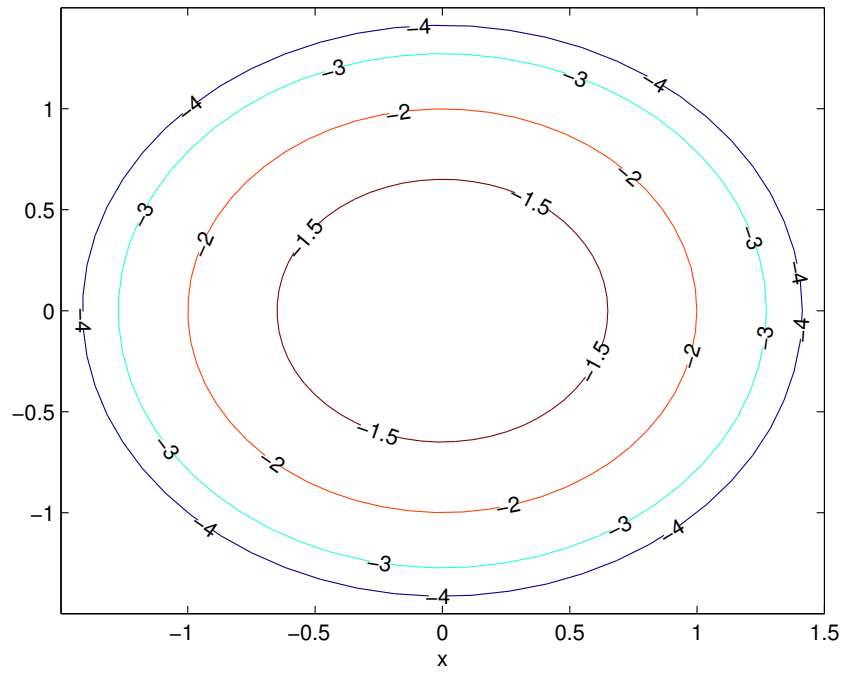


Figure 4.2: We show the level sets of $L_m(\mathbf{x}) = b$ for $b = -1.5, -2, -3, -4$.

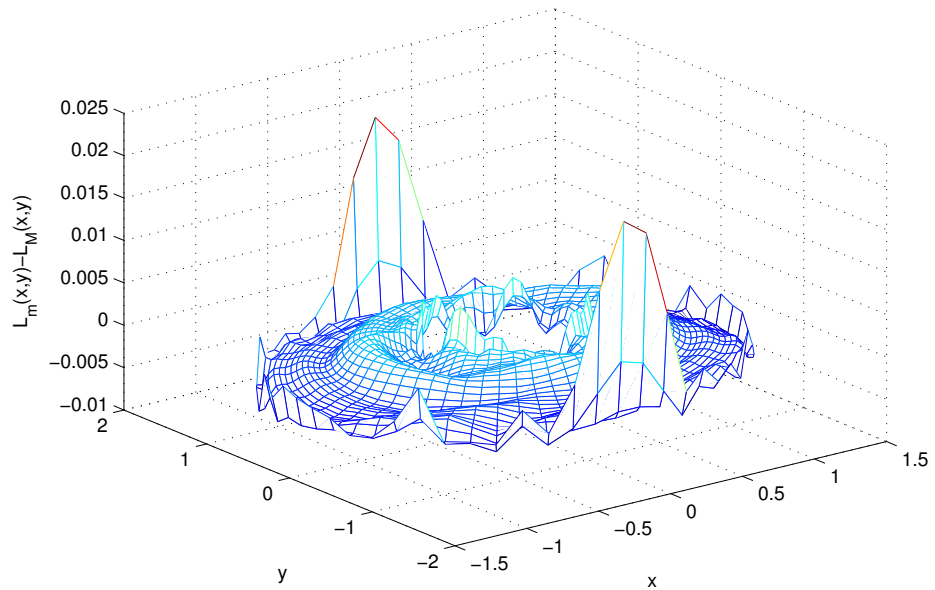


Figure 4.3: Shows $L_m(\mathbf{x}) - L_M(\mathbf{x})$. The relatively high errors on the boundary around $x = 0$ are due to sparser grid in that region.

4.5.2 Numerically based Example

We compare our method to the numerical results given in [Giesl, 2007b], in that paper the author shows their method working on the Van-der-Pol equation $\ddot{x} + (x^2 - 1)\dot{x} = 0$, which the author shows becomes,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + (1 - x^2)y. \end{cases} \quad (4.64)$$

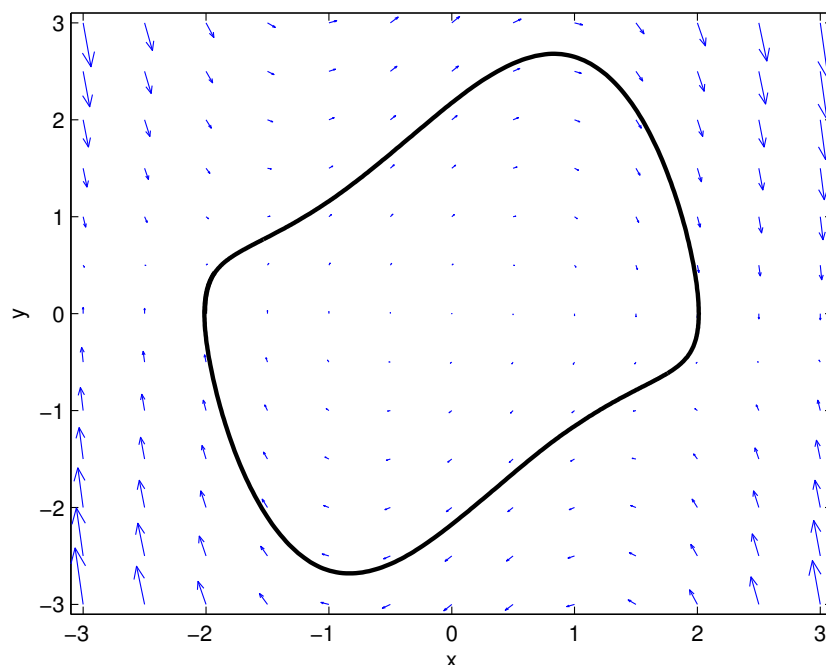


Figure 4.4: A quiver plot of the Van-der Pol system (4.64) and its periodic orbit (black).

We show the periodic orbit and phase diagram for this system in Figure 4.4. The periodic orbit was numerically approximated using a standard function in Matlab. We build our approximant to the system (4.64) as described in Section 4.3. To approximate W we use the Wendland function $\Psi(\mathbf{x}) = \psi_{6,4}(\|\mathbf{x}\|)$ which is defined in Table 2.2. Our grid consists of 3583 points, the hexagonal grid separates points by 0.1, thus $h = 0.1 \frac{\sqrt{5}}{2}$. The support radius of the Wendland function was 1. The grid points and the level set $L_m(\mathbf{x}) = 0$ is shown in Figure 4.5. We note that we had to use significantly more grid points than

the method in [Giesl, 2007b]. This is mitigated slightly by the smaller support radius (1 vs 5.5). Also in [Giesl, 2007b] an adaptive grid is used where we have used a uniform grid. A deeper comparison of the numerical demands of the two methods warrants further study. In Figure 4.6 we highlight the set $\{\mathbf{x} \in \mathbb{R}^2 \mid L_m(\mathbf{x}) > 0\}$, if we can find a positively invariant set which coincides with set $\{\mathbf{x} \in \mathbb{R}^2 \mid L_m(\mathbf{x}) \leq 0\}$, will have determined a subset of the basin of attraction of Ω .

To determine the positively invariant set K we follow the method given in [Giesl, 2007b] as closely as possible (the grid placement was not exactly replicated). A brief overview of the main theorems concerning the numerical determination of a positively invariant set are given in Section 4.6 for further details see [Giesl, 2007b]. We want to approximate the solution of the equation $V'(\mathbf{x}) = -1$. We use the Wendland function $\Psi(\mathbf{x}) = \psi_{5,3} \left(\frac{1}{5.5} \|\mathbf{x}\| \right)$, where $\psi_{5,3}(r) = (1-r)_+^8 [32r^3 + 25r^2 + 8r + 1]$. We used 602 grid points to build the approximation. In Figure 4.7 the grid points are shown, furthermore the red set identifies the level set $v'(\mathbf{x}) = 0$. In Figure 4.8 we see that there are small areas near the periodic orbit where $v'(\mathbf{x}) > 0$, this is a known problem with the method we employed for determining the positively invariant set. In Figure 4.8 we also see the set $K = \{\mathbf{x} \in \mathbb{R}^2 \mid v(\mathbf{x}) \leq -0.5\}$ (blue), this is compact and connected set that does not contain an equilibrium that is fully contained within the set where $v'(\mathbf{x}) < 0$. Hence K is a positively invariant set. Finally Figure 4.9 shows the positively invariant set K along with the set where $L_m(\mathbf{x}) \leq 0$ (magenta). Therefore the set K contains a unique exponentially stable periodic orbit Ω and $K \subset A(\Omega)$. Although the method works it did require considerably more data points than the method proposed in [Giesl, 2007b], however the method proposed in this chapter did not rely on any attributes of the periodic orbit being known to be determine the approximation.

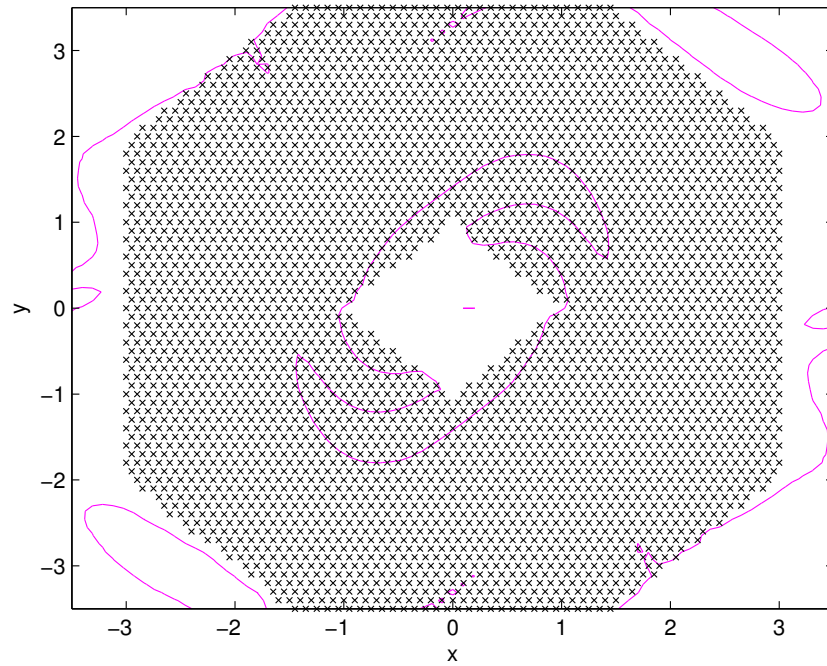


Figure 4.5: The grid for the approximation of W and the level set $L_m(\mathbf{x}) = 0$ (magenta) for the Van-der Pol system given in (4.64).

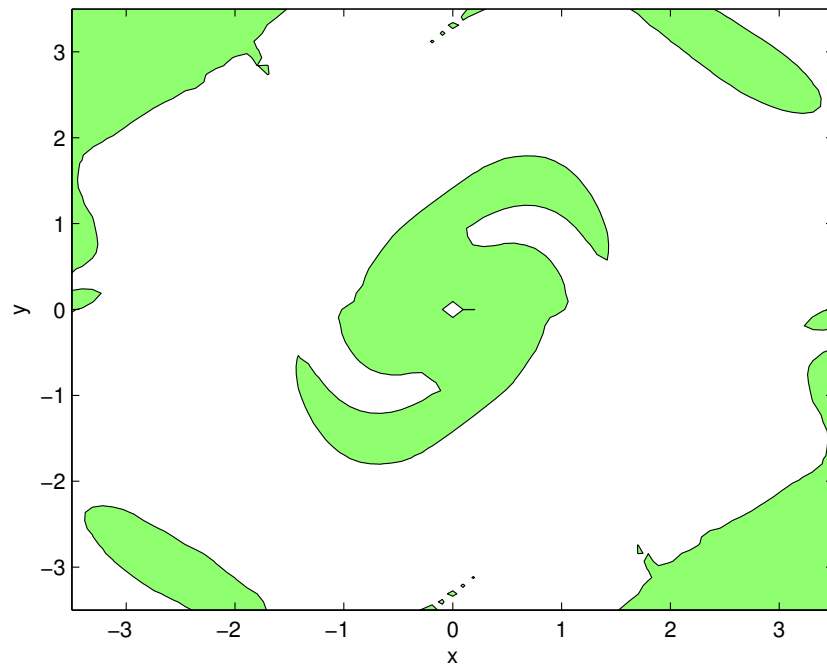


Figure 4.6: The green area denotes the set $\{\mathbf{x} \in \mathbb{R}^2 \mid L_m(\mathbf{x}) > 0\}$.

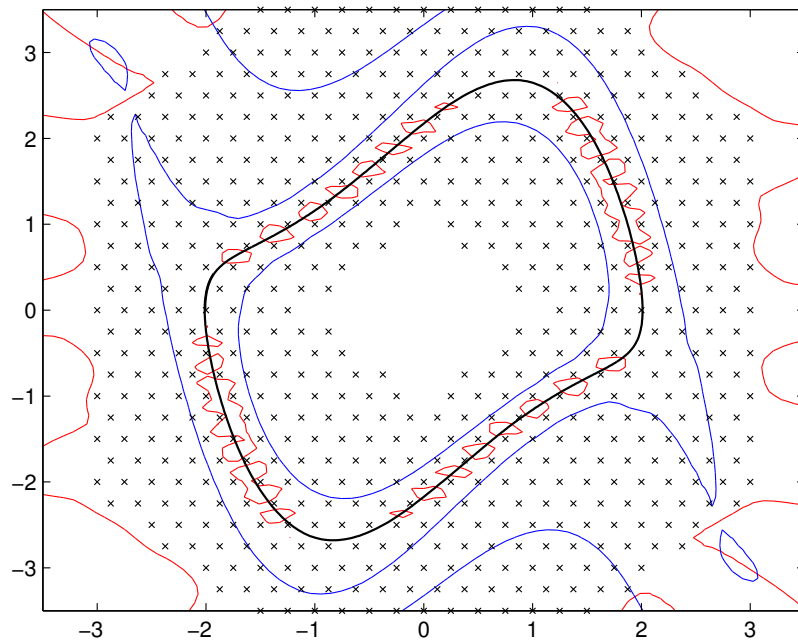


Figure 4.7: The figure shows the grid points used to build v , the approximation of function V defined in Theorem 4.7.1, the level set $v'(\mathbf{x}) = 0$ (red) and the sublevel set $K = \{\mathbf{x} \in \mathbb{R}^2 \mid v(\mathbf{x}) \leq -0.5\}$ (blue).

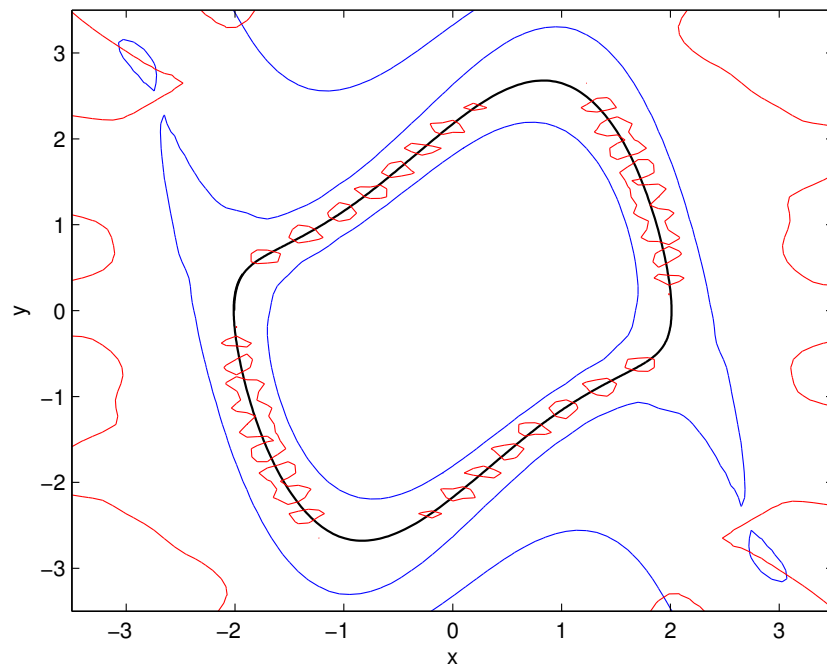


Figure 4.8: The figure shows the periodic orbit of the system (black), the level set $v'(\mathbf{x}) = 0$ (red) and the sublevel set $K = \{\mathbf{x} \in \mathbb{R}^2 \mid v(\mathbf{x}) \leq -0.5\}$ (blue).

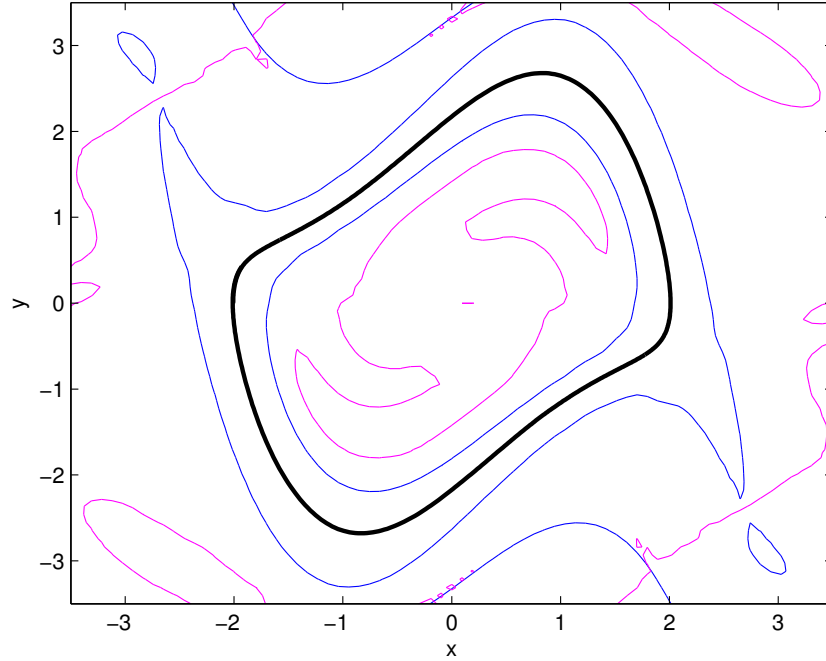


Figure 4.9: The figure shows the periodic orbit of the system (black), the positively invariant set $K = \{\mathbf{x} \in \mathbb{R}^2 \mid v(\mathbf{x}) \leq -0.5\}$ (blue) and the level set $L_m(\mathbf{x}) = 0$ (magenta).

4.6 Appendix: Determining a Positively Invariant Set

Corollary 4.3.1 gives a partial differential equation that can be used to determine an approximate of W . The corollary requires a positively invariant set. Hence before approximating a subset of $A(\Omega)$ it is necessary to determine a positively invariant set. A methodology to do this laid out in Section 3 of [Giesl, 2007b], we give a brief overview of this approach below.

Proposition 4.7. *Let $V \in C^1(\mathbb{R}^2, \mathbb{R})$. Let $S := \{x \in \mathbb{R}^2 \mid V(x) \leq \rho\}$ be non-empty. Let $V'(x) < 0$ hold for all x with $V(x) = \rho$. Then S is positively invariant.*

The proof can be found in [Giesl, 2007b]. A function satisfying the conditions of Proposition 4.7 is shown to be satisfied by Theorem 3.2 of [Giesl, 2007b]. This is done by finding the level set of a constant Lyapunov function such as that defined in Theorem 2.38 of [Giesl, 2008]. The method is similar in its approach to the one showing the existence of a non-characteristic surface that is used to establish the global existence of W in The-

orem 4.2.1. Directly approximating a function of the form V in Proposition 4.7 would be difficult as the level set on which the orbital derivative of V is defined on is unknown. Proposition 4.7 circumvents this problem by showing the existence of a function satisfying $V'(x) = -c < 0$ for all $x \in A(\Omega) \setminus \Omega$. It can be seen that as long as ρ is chosen such that $S \subset A(\Omega)$ then $V'(x) < 0$ on ∂S and hence S will be positively invariant.

Theorem 4.7.1. *Consider $\dot{x} = f(x)$, where $f \in C^\sigma(\mathbb{R}^2, \mathbb{R}^2)$, $\sigma \geq 2$. Let Ω be an exponential stable periodic orbit with Floquet exponents 0 and $-\nu$, let $c > 0$.*

Then there is a function $V \in C^{\sigma-1}(A(\Omega) \setminus \Omega, \mathbb{R})$ such that,

$$V'(x) = -c \text{ for all } x \in A(\Omega) \setminus \Omega.$$

It should be noted that V is not defined on Ω , this is because it approaches $-\infty$ as $\text{dist}(x, \Omega) \rightarrow 0$, this does mean that when V is approximated there should not be any data points on Ω . The approximation of such a problem with meshless collocation is shown in Section 2.7.

Chapter 5

Discussion

The work began by outlining its contribution to the field and giving some of the theoretical foundations of meshless collocation, Borg's criterion and dynamical systems. To give context to the work's place in the existing literature the current state of the art was then discussed for areas of research that this work touches upon.

The major contributions of this work can be found in Chapters 3 and 4, which show how to approximate areas of exponential attraction for finite-time dynamical systems and a method to approximate the basin of attraction of a periodic orbit in \mathbb{R}^2 respectively. Both chapters follow a similar pattern. They both introduce the problem, provide some background, and then go on to show how the derivation of some existing results can be simplified in the restricted spatial dimensions. They then show the derivation of the collocation equation. The orbital derivative of the collocation equation is taken to eliminate some unknown values. We then present numerical schemes to approximate the weight functions that are the solution of these reduced partial differential equations. It is shown that both numerical schemes are convergent. Finally, some examples show the application of the numerical schemes.

The numerical schemes introduced to approximate attractive sets in this work have the advantage that they do not require a-priori knowledge of the solutions or periodic orbit before they are applied. This is due to their use of contraction metric methods that, unlike Lyapunov methods, do not require a-priori knowledge of the equilibrium or invariant set that is of interest. However, when deriving the collocation equations that are used to approximate the weight function, unknown elements are introduced that depend on the

rate of exponential attraction of solutions (in the finite-time case) or the average velocity of the solution on the periodic orbit (in the periodic orbit case). These elements initially prevented the numerical scheme from being implemented without a-priori knowledge of the invariant sets. However as these elements are constant along solutions it is possible to eliminate them from the relevant equations by taking a second orbital derivative. With these elements removed from the equations it is possible to apply meshless collocation and construct an approximant of the weight function. The extra application of the orbital derivative does add complexity to the derivation of the components of the collocation equation, however, these results are shown in this work.

We then go on to show that the numerical scheme on both domain types is convergent. The convergence proof is simpler in the case of the finite-time domains as the collocation problem is a boundary value problem. The boundary values can be used when integrating along solutions. In the case where the scheme is used to approximate the basin of attraction of periodic orbits, we do not have an explicit boundary problem to aid with the integration along solutions. However the periodic orbit, Ω , is used as a surrogate boundary condition as we know that W will be periodic on the orbit with a period T . We then use the result on the periodic orbit to show that the scheme is convergent in a small neighbourhood of Ω . Finally we use the exponential stability of the orbit to show that all points in the basin of attraction will reach the neighbourhood of Ω within a finite time and thus show that the scheme is convergent.

One area of further study would be a deeper exploration of the application of the numerical schemes to examples. In the finite-time case it would be interesting to study how the approximation of the area of exponential attraction can be used along with the Lyapunov method to approximate the domain of attraction of a solution. For the scheme to approximate the basin of attraction of a periodic orbit it would be interesting to compare the numerical efficiency of the method introduced in this work with the method introduced in [Giesl, 2007b], which requires a-priori information about the periodic orbit to be implemented. Another area of further study would be to extend the scheme to higher spatial dimensions, although the method to do this is not immediately clear. The first difficulty is that the extended spatial dimensions will mean that the existing method of replacing the Riemannian metric with a scalar weight function will not, in general, be possible. Also in

higher dimensions the methods described in this chapter to derive the collocation equation will deliver an inequality rather than an equation. Therefore further work will be required to understand how to extend this work into higher dimensions.

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